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ABSTRACT

This is part one of a two-part SMSG elementary school text for fourth-grade students. The content is aimed at the development of some of the fundamental concepts of mathematics such as number, numeration, the operations of arithmetic, and intuitive geometry. Chapter topics include concept of sets, numeration, properties and techniques of addition and subtraction, properties of multiplication and division, and sets of points. (MP)

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GRADE 4
PART I

SE 027 908

SCHOOL MATHEMATICS STUDY GROUP

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School Mathematics Study Group

Mathematics for the Elementary School, Grade 4

Unit 25

Mathematics for the Elementary School, Grade 4

Student's Text, Part I

REVISED EDITION

Prepared under the supervision of the
Panel on Elementary School Mathematics
of the School Mathematics Study Group:

Leslie Beatty	Chula Vista City School District, Chula Vista, California
E. Glenadine Gibb	Iowa State Teachers College, Cedar Falls, Iowa
W. T. Guy	University of Texas
S. B. Jackson	University of Maryland
Irene Sauble	Detroit Public Schools
M. H. Stone	University of Chicago
J. F. Weaver	Boston University
R. L. Wilder	University of Michigan

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

As one of its contributions to the improvement of mathematics in the schools of this country, the School Mathematics Study Group has prepared a series of sample text materials for grades 4 through 6. These are designed to illustrate a kind of mathematics curriculum that we believe appropriate for elementary schools.

This volume is a portion of these materials which were prepared by a group of 30 individuals, divided almost equally between distinguished college and university mathematicians and master elementary teachers and consultants. A strong effort has been made on the part of all to make the content of this text material mathematically sound, appropriate and teachable. Preliminary versions were used in numerous classrooms both to strengthen and to modify these judgments.

The content is designed to give the pupil a much broader concept, than has been traditionally given at this level, of what mathematics really is. There is less emphasis on rote learning and more emphasis on the construction of models and symbolic representation of ideas and relationships from which pupils can draw important mathematical generalizations.

The basic content is aimed at the development of some of the fundamental concepts of mathematics. These include ideas about:

number; numeration; the operations of arithmetic; and intuitive geometry. The simplest treatment of these ideas is introduced early. They are frequently re-examined at each succeeding level and opportunities are provided throughout the texts to explore them more fully and apply them effectively in solving problems. These basic mathematical understandings and skills are continually developed and extended throughout the entire mathematics curriculum, from grades K through 12 and beyond.

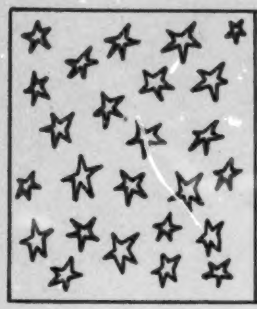
We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that these texts may greatly assist all pupils and teachers who use them to achieve this goal, and that they may experience something of the joy of discovery and accomplishment that can be realized through the study of mathematics.

Chapter 1

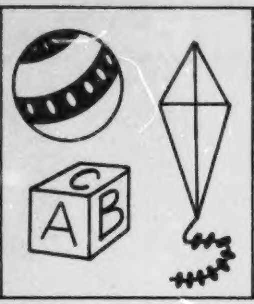
CONCEPT OF SETS

THINKING ABOUT SETS

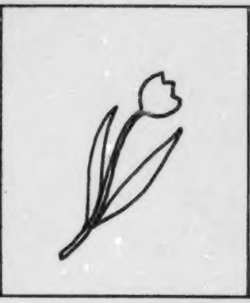
These are pictures of sets.



A Set of Stars



A Set of Toys



A Set of Flowers

An Empty Set

You can think of many sets of things--

The set of children in your school;

The set of children in your class;

The set of numbers, 1, 2, 3, 4, 5, and so on;

The set of numbers, 1, 3, 5, 7, 9, 11, and so on;

The set of numbers, 2, 4, 6, 8, 10, 12, and so on;

The set of letters in the alphabet;

The set of boys in your class who are ten feet tall.

A set is a collection of things. Some of these collections can be sets of objects, sets of people, sets of pictures, and sets of numbers. Think of some examples of sets of things.

A thing that belongs to a set is a member of that set. Each of the letters, b, r, s, t, y, is a member of the set of letters in our alphabet. You are a member of the set of children in your school.

There are sets that have only one member. The set of letters in our alphabet between d and f has only one member. It is the letter e.

There are sets that have no members. The set of children in your class, who are less than four years old, has no members. If a set has no members, it is called the empty set.

We use capital letters for names of sets.
You may use any capital letter you wish.
The letter you choose may help you remember the set.
The states New York and California are members of the set of states of the U. S. A.
We may call this set, Set C. We write

$$C = \{\text{New York, California}\}$$

The counting numbers between 4 and 8 are 5, 6, 7.
We may call this set, Set N. We write:

$$N = \{5, 6, 7\}$$

Exercise Set 1

Name the members of each set:

1. The first five letters of the alphabet.
2. The numbers that you use when you count the first five children in your classroom.
3. The numbers counting by 2's, beginning with 1 and ending with 9.
4. The numbers counting by 2's, beginning with 6 and ending with 16.
5. The letters in your first name.
   (A letter may appear in your first name more than once. Use it only once in the set.)
6. The days of the week whose names begin with "M".
7. The boys in your class less than six years old.
8. The months of the year whose names begin with letter J.
9. The numbers between 30 and 40 that are larger than 50.
10. **BRAINTWISTER:** The letters which are in the name of your school and not in your last name.

NUMBERS

When you first learned to count, you began with 1. You counted 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and so on. You can count much farther than 12 now. No matter how far you can count, there are still more numbers. If you knew how to count them, you could keep on counting as long as you live. Then, there would still be more numbers. These numbers used in counting are called counting numbers.

In arithmetic there is the set of numbers called the set of whole numbers. These numbers are 0, 1, 2, 3, 4, 5, 6, and so on. We may write the set of whole numbers this way:

$$W = \{0, 1, 2, 3, 4, 5, 6, . . .\}$$

We may write the set of counting numbers this way:

$$C = \{1, 2, 3, 4, 5, 6, . . .\}$$

We cannot write all the whole numbers. We use the three dots, ... to mean that there are more numbers than we can write.
The number 0 is the first one written in Set W.
The number 6 is the last number written in the Set W.
But, the number 6 is not the last whole number.
It is just the last number written in Set W.

We write:
$W = \{0, 1, 2, 3, ...\}$
$W = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...\}$
We have used two different ways to name the same set.

Count "by 2's" beginning with the number 0.
You count 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, . . .
These numbers that you name are called even numbers.
The numbers 0, 2, 4, 6, 8, . . ., are called even numbers.
The numerals 38, 54, 76, 128, 100, 200, 1352, are names of some even numbers.

Count "by 2's" beginning with the number 1.
You count 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, . . .
The whole numbers that you name when you count "by 2's" beginning with 1 are called odd numbers.
The numerals 25, 37, 41, 53, 101, 421, 1247, are names of some odd numbers.

Here are some more sets of things.

Mary	a
tree	b
pen	c
car	d
picture	
Set A	Set B

Set A is a set of words.
The number of words in Set A is 5.

Set B is a set of letters.
The number of letters in Set B is 4.

The number of odd numbers in the set of counting numbers between 1 and 20 is 9. The members of this set are 3, 5, 7, 9, 11, 13, 15, 17, 19.

The number of words in the set { } is 0.
There are no members of this set.
We call this set the empty set.
The number of the empty set is zero.

We use numbers to tell how many members are in a set.

Exercise Set 2

Here are some sets in 1, 2, 3, 4, and 5. Below each set are groups of words. Which best describes each set? Is it a), b), or c)? Write your answer. Then tell how many members are in that set.

1. {1, 3, 5, 7, 9}
   - a) a set of small numbers
   - b) the set of all odd numbers
   - c) the set of odd numbers less than 10

2. {Tuesday, Thursday}
   - a) the set of school days
   - b) the set of the last two days in the week
   - c) the set of days in the week whose names begin with T

3. {10, 20, 30, 40}
   a) the set of numbers less than 50
   b) the counting numbers less than 50 whose numerals end in zero
   c) the set of even numbers less than 50

4. {chalk, book, eraser, pencil}
   a) a set of things you find in a schoolroom
   b) a set of school furniture
   c) a set of things to read

5. {bus, train, automobile, airplane}
   a) a set of things you see in the sky
   b) a set of things you find in a garage
   c) a set of things people may use when they travel

6. Here are some things: potato, 9, Bobby, celery, 3, rock, 5,
George, 15, e, 4, bacon, 6, Mary, u,
David, a, candy, o, 7, 1, key

Select the things that are:
   a) a set of boys' names
   b) the set of whole numbers larger than 2 and less than 8
   c) the set of vowels
   d) a set of things to eat
   e) a set of things to read

SETS WITHIN SETS

We had some coins in a piggy-bank.

We poured them out on the table.

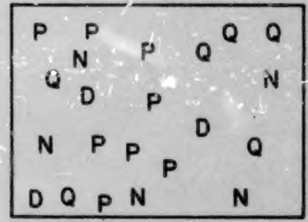
This picture shows the way they fell.

Each N shows a nickel.

Each P shows a penny.

Each D shows a dime.

Each Q shows a quarter.

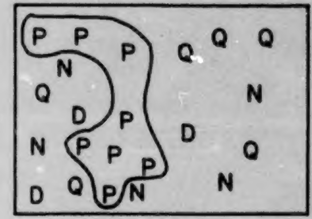

P P Q Q
N P Q
Q N
D P
D
N P P Q
P
D Q P N N

In the set of coins there is a set of pennies.

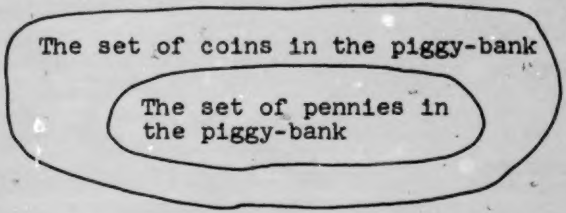
A fence is around all the pennies.

All pennies are inside the fence.

All other coins are outside the fence.


P P Q Q Q
P
N
Q N
D P
D
N P P Q
P
D Q P N N

There is another way to show that the set of pennies is within the set of coins. We can show it like this.

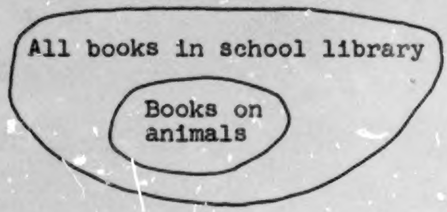

The set of coins in the piggy-bank
The set of pennies in the piggy-bank

How do we show the pennies in the picture?

They are inside the small ring.

Where are the other coins that are not pennies?

They are outside the small ring but inside the big ring.


All books in school library
Books on animals

This picture shows another set within a set. The set of books on animals is within the set of all books in the school library.

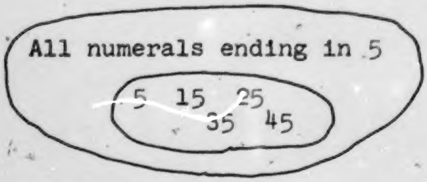
We can say that the set of books on animals is a subset of the set of all books in the library.

Exercise Set 3

1. The set of all pupils in your school is within the set of all pupils in your state. Draw a picture to show this idea.

2. **Make a drawing** to show that the set of all even numbers is within the set of all whole numbers.

3. The drawing below shows that the set of numerals 5, 15, 25, 35, 45 is within the set of all numerals ending in 5.


All numerals ending in 5
5 15 25
35 45

Make a drawing to show that the numbers 10, 20, 30, 40 are within the set of all whole numbers.

4. A set of girls in the fourth grade is Mary, Martha, Karen, Kathy, Marian, Sue. Call this set, Set S.

S = {Mary, Martha, Karen, Kathy, Marian, Sue}

Here is some information about this set of girls. Use it in answering the questions in this problem.

Name	Color of Eyes	Color of Hair	Age
Mary	blue	blonde	9
Martha	brown	brown	10
Karen	gray	black	9
Kathy	brown	black	9
Marian	blue	brown	10
Sue	brown	brown	9

a) Write the members of the set of girls who are 10 years old. Call this set, Set B. Is Set B in Set S?

b) Write the members of the set of girls who have gray eyes. Call this set, Set C. Is Set C in Set S?

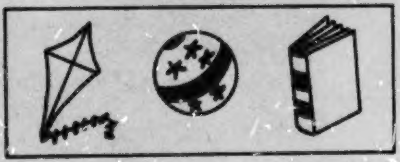
c) Write the members of the set of girls who have black hair and who are 9 years old. Call this set, Set X. Is Set X in Set S?

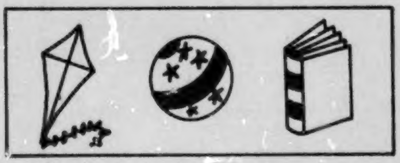
d) Is Set C in Set X?

5. **BRAINTWISTER:** Make a drawing to show that the numbers 2, 4, 6, 8, 10 are within the set of even numbers and the even numbers are within the set of whole numbers.

EQUAL SETS

Here are two sets of pictures.





The members of the two sets are the same.

If two sets have the same members, the two sets are equal.

The members of equal sets do not have to be in the same order.

Here are some other sets.

A = {apple, pencil}

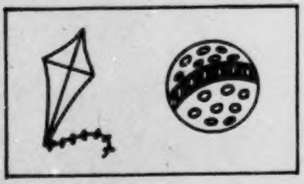
B = {pencil, apple}

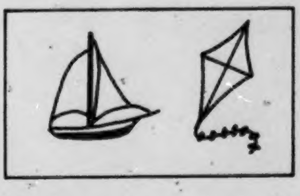
We can say that Set A equals Set B. We write: Set A = Set B

M = {5, 1, 3}

N = {1, 3, 5}

Does Set M = Set N? Why?





Here are two sets of pictures.

These sets do not have the same members.

They do have the same number of members.

G = {apple, pencil, house}

H = {dog, car, hat}

Set G is not equal to Set H. We write: Set G ≠ Set H

R = {0, 1, 2, 3}

P = {1, 2, 3}

Set R is not equal to Set P. We write: Set R ≠ Set P.

Exercise Set 4

A = {4, 5, 7}   B = {5, 4, 7}   C = {7, 4, 5}

1. Does Set A = Set B?
2. Does Set C = Set A?
3. Does Set C = Set B?

X = {b, a, c, k}   Y = {c, b, k, a}   Z = {k, c, t, a}

4. Does Set X = Set Y?
5. Does Set Z = Set X?
6. Does Set Z = Set Y?

R = {6, 10, 8}   S = {10, 7, 8}   T = {8, 6, 10}

7. Does Set R = Set S?
8. Does Set T = Set R?
9. Does Set T = Set S?

Here are some sets: (Use these to answer questions 10, 11, 12, 13.)

A = {3, 5, 7, 4, 2}
B = {2, 3, 4, 5, 6}
C = {2, 3, 4, 5, 7}
D = {6, 5, 4, 3, 2}
E = {5, 3, 7, 2, 4}
F = {2, 4, 6, 3, 5}

10. Set A is equal to what sets?

11. Set C is equal to what sets?

12. Are Set C, Set E, and Set A equal sets?

13. Which sets are equal to Set D?

14. X = {t, s, r, d}.

Think of a set that has the same number of members as Set X but is not equal to Set X.

Call it Set Z.

Copy and finish: Z = {   }.

15. B = {3, 7, 9, 5}.

Set E is the set of all odd numbers less than 10.

Which is correct? Set B = Set E or Set B ≠ Set E.

16. Set A is the set of all whole numbers greater than 5 but less than 10.

Set B is equal to Set A.

Name the members of Set B.

17. $D = \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$

$E = \{\frac{1}{4}, \frac{3}{4}, \frac{1}{2}\}$

Which is correct? Set $D \neq$ Set E or Set $D =$ Set E.

THE UNION OF SETS

John took a hike with his mother and father.

John kept a record of the different birds his mother saw.

He kept a record of the different birds his father saw.

Set A is the set of different birds John's mother saw.

robin crow sparrow

Set A

Set B is the set of different birds John's father saw.

hawk bluejay
wren eagle

Set B

To find all the different birds John's parents saw, we put Set A and Set B together. Our set is now

robin, crow, sparrow, hawk, wren, bluejay, eagle

This set is the union of Set A and Set B.

We write:

$A \cup B = \{$robin, crow, sparrow, hawk, wren, bluejay, eagle$\}$

We read $A \cup B$: the union of Set A and Set B.

Your class chose some committees for a party.

The committee to select the games was Set G.

The committee to buy the prizes was Set P.

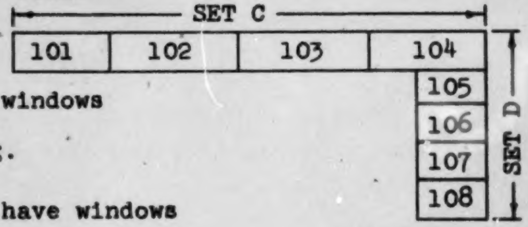
$G = \{$John, James, Helen, Susan$\}$

$P = \{$John, Irene, Phyllis, Samuel$\}$

The two committees met together. What pupils attended the meeting?

$G \cup P = \{$John, James, Helen, Susan, Irene, Phyllis, Samuel$\}$

The picture at the right shows rooms in Jane's school.



Rooms 101, 102, 103, 104 have windows along one side of the building.

Rooms 104, 105, 106, 107, 108 have windows along another side of the building.

$$C = \{101, 102, 103, 104\}$$

$$D = \{104, 105, 106, 107, 108\}$$

We write: $C \cup D = \{101, 102, 103, 104, 105, 106, 107, 108\}$

We read: The union of Set C and Set D is the set whose members are 101, 102, 103, 104, 105, 106, 107, 108.

Eddie is learning to play the trumpet and the piano.

His practice schedule looks like this.

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
		TRUMPET	TRUMPET	TRUMPET	TRUMPET	
	PIANO	PIANO	PIANO	PIANO		

Set T is the set of days that Eddie practices the trumpet.

Set P is the set of days that Eddie practices the piano.

$$T = \{\text{Tuesday, Wednesday, Thursday, Friday}\}$$

$$P = \{\text{Monday, Tuesday, Wednesday, Thursday}\}$$

The union of Set T and Set P is the set of days in the week when Eddie practices the trumpet or the piano or both.

We write:

$$T \cup P = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday}\}$$

Exercise Set 5

1. A = {cat, dog, cow, horse}

B = {duck, horse, pig}

Which one of these sets is the union of Set A and Set B?

X = {cat, cow, dog, duck, horse, pig}

Y = {cow, horse, duck, horse, pig}

Z = {cat, dog, cow, hen, duck, horse, pig}

Answer. Set X is the union of Set A and Set B. We write

$A \cup B = X$

2. R = {10, 20, 30, 40, 50}

S = {60, 70, 80, 90, 100}

Which one of these sets is the union of Set R and Set S?

M = {70, 90, 110, 130, 150}

N = {100, 90, 80, 70, 60, 50, 40, 30, 20, 10}

Copy and finish: $R \cup S =$

3. G = {a, t, z, r, m, j}

H = {r, q, z, t}

Which one of these sets is the union of Set G and Set H?

M = {q, a, m, j}

N = {a, t, z, r, m, j, q, r, z, t}

L = {a, j, m, q, r, t, z}

Copy and finish: $G \cup H =$

4. $J = \{white, blue\}$

$K = \{red, blue\}$

Copy and finish: $J \cup K =$

5. $V = \{18, 21, 24\}$

$W = \{15, 18, 21, 24, 27\}$

Copy and finish: $V \cup W =$

6. $N = \{s, o, a, p\}$

$O = \{w, a, t, e, r\}$

Copy and finish: $N \cup O =$

7. Set P is the set of odd numbers between 6 and 12. Copy and finish: $P =$

Set Q is the set of odd numbers less than 7. Copy and finish: $Q =$ and $P \cup Q =$

8. Set R is the set of even numbers between 90 and 100. Copy and finish: $R =$

Set S is the set of whole numbers greater than 94 and less than 96.

Copy and finish: $S =$ and $R \cup S =$

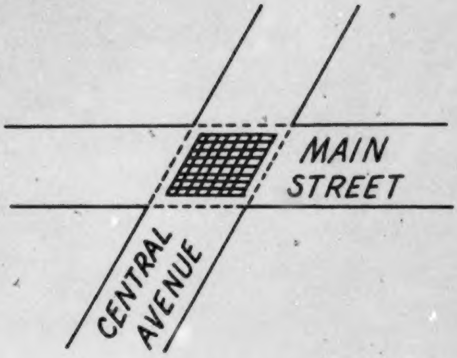
9. Set T is the set of whole numbers between 65 and 66. Copy and finish: $T =$

Set W is the set of whole numbers larger than 9 and less than 11. Copy and finish: $W =$ and $T \cup W =$

10. Set X is the set of counting numbers between 25 and 30. Copy and finish: $X =$

Set Y is the set of even numbers between 25 and 31. Copy and finish: $Y =$ and $X \cup Y =$

THE INTERSECTION OF SETS



Look at the picture at the right.
Main Street and Central Avenue cross each other. A part of one street is also a part of the other street.
It has been shaded in the picture.
This part belongs to both streets.
It is the intersection of the two streets.

Look at these two sets:

Set A	Set B
Alice	Ellen
Betty	Ken
Ken	Joe
Sue	Sue
Tom	Wendy

Some children are members of both sets.
The children who are members of both sets are Ken and Sue.
This set may be written {Ken, Sue}.
This set is called the intersection of Set A and Set B.
We write: $A \cap B = \{\text{Ken, Sue}\}$.
We read $A \cap B$: the intersection of Set A and Set B.
The symbol $\cap$ means "the intersection of."

Here are some more sets:

Set X is the set of numbers we use when we count by fives, starting with 5 and ending with 30.

$$X = \{5, 10, 15, 20, 25, 30\}$$

Set Y is the set of numbers we use when we count by tens, starting with 10 and ending with 50.

$$Y = \{10, 20, 30, 40, 50\}$$

The numbers that are members of both sets X and Y are 10, 20, and 30.

The intersection of Set X and Set Y is the set $\{10, 20, 30\}$.
We write: $X \cap Y = \{10, 20, 30\}$.

$$J = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$$
$$K = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

Set J is the set of even numbers less than 17.
Set K is the set of odd numbers less than 17.

There are no numbers that are members of both Set J and Set K.
The intersection of Set J and Set K is the set $\{\ \}$.

We write: $J \cap K = \{\quad\}$.

Exercise Set 6

1. A = {car, train, taxi, boat}.

   B = {wagon, boat, airplane, train, bicycle}.

   Which one of these sets is the intersection of Set A and Set B?

   M = {car, taxi, wagon, airplane, bicycle}

   R = {boat, train}

   S = { }

   Answer. Set R is the intersection of Set A and Set B. We write $A \cap B = R$.

2. D = {13, 17, 19, 23}

   E = {9, 11, 13, 15, 17, 18, 21}

   Which one of these sets is the intersection of Set D and Set E?

   P = {13, 17, 19, 23, 9, 11, 13, 15, 17, 19, 21}

   Q = {9, 11, 15, 19, 21}

   R = {17, 13}

   Copy and finish: $D \cap E =$

3. G = {d, x, p, r, q, m}

   H = {t, b, s, n, a}

   Which one of these sets is the intersection of Set G and Set H?

   I = {a, b, d, m, n, p, q, r, s, t, x}

   J = {d, b, p, q, n, m}

   K = { }

   Copy and finish: $G \cap H =$

4. $J = \{\text{dress, shoe, hat, coat}\}$

$K = \{\text{shoe, cap, coat, dress}\}$

Copy and finish: $J \cap K =$

5. $L = \{g, r, a, n, d\}$

$M = \{p, i, a, n, o\}$

Copy and finish: $L \cap M =$

6. $N = \{73, 59, 8, 81, 63\}$

$O = \{104, 49, 73, 58, 18, 95\}$

Copy and finish: $N \cap O =$

7. Set P is the set of whole numbers less than 7.

Copy and finish: $P =$

Set Q is the set of whole numbers between 5 and 12.

Copy and finish: $Q =$

Copy and finish: $P \cap Q =$

and $P \cup Q =$

8. Set R is the set of whole numbers larger than 38 and less than 44.

Copy and finish: $R =$

Set S is the set of numbers between 36 and 46 that are not even numbers.

Copy and finish: $S =$

Copy and finish: $R \cap S =$

$R \cup S =$

SUPPLEMENTARY PRACTICE EXERCISES

THINKING ABOUT SETS -- Exercise Set 7

Write the members of each of these sets.

1. The set of even numbers less than 12.

2. The set of counting numbers less than 20 and larger than 10.

3. The set of odd numbers between 10 and 2[illegible].

4. The set of whole numbers less than 17 and larger than 15.

5. The set of numbers between 30 and 40 that are larger than 60.

6. How many members are there in the set of letters of our alphabet?

7. Here is a set: {Tuesday, Thursday}. Describe this set by writing on your paper: This is a set of ________________ ____.

8. Name two sets that have no members.

9. Make a picture of a set. Then describe the set by saying: This is a set of:________________________________.

10. Describe this set in your own words:

$$A = \{5, 10, 15, 20, 25\}$$

SETS WITHIN SETS -- Exercise Set 8

1. Draw a picture to show that the set of dimes is a set within the set of United States coins.

2. A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   Set B = the set of all counting numbers.

   Draw a picture to show that Set A is a set within Set B.

   Make up one subset (set within a set) for each of the following:

3. {a, b, c, d, e, f}

4. {Ford, Chevrolet, Plymouth, Rambler, Cadillac}

5. {The set of holidays in a year}

6. Write the set of vowels. Call this set Set A. Now write a subset of Set A.

7. Set C = the set of all states in the United States.
   Write a subset of Set C.

8. Set X = the set of dimes. We can say Set X is a set within the set of all ________________________________.

9. Set Y = the set of counting numbers.
   Z = {3, 8, 15, 93, 175}
   Set Z is a ______________ of Set Y.

10. Set C is the set of all odd numbers. Make up a subset of Set C.

EQUAL SETS -- Exercise Set 9

1. A = {9, 4, 3}   B = {4, 3, 9}   Does Set A = Set B?

2. M = {10, 11, 12}   N = {10, 11, 13}
   Does Set M = Set N?

3. X = {dog, cat, mouse}   Y = {mouse, cat, horse, dog}
   Does Set X equal Set Y?

4. Set D is the set of whole numbers greater than 7 and less than 12. Set E is equal to Set D. Name the members of Set E.

Here are some sets for exercises 5-7.

F = {2, 4, 6, 8}
G = {4, 2, 8, 6}
H = {8, 6, 1, 4}
K = {8, 6, 1, 2}
L = {4, 8, 6, 1}
M = {6, 2, 8, 4}

5. Set F is equal to what sets?

6. Are Set H and Set L equal sets?

7. Which sets are equal to Set K?

8. A = {2, 4, 6, 8} Set B is the set of all even numbers less than 20. Which is correct? Set A = Set B or Set A ≠ Set B?

9. Make up a set. Call this set Set X. Now make up a set that is equal to Set X.

10. Set J is the set of the first five letters of the alphabet.

Set K is the set of the last five letters of the alphabet.

Does Set J = Set K?

THE UNION OF SETS -- Exercise Set 10

1. A = {1, 2, 3, 4} B = {5, 6, 7, 8}

   The union of Set A and Set B is the set:

2. C = {c, a, n, d, y} D = {c, o, k, e}

   Copy and finish: $C \cup D =$

3. E = {5, 10, 15, 20, 25} F = {30, 35, 40, 45}

   Which one of these sets is the union of Set E and Set F?

   M = {10, 20, 25, 45, 50, 55, 60, 65, 70}

   N = {45, 40, 35, 30, 25, 20, 15, 10, 5}

4. H = {2, 4, 6, 8, 10} J = {3, 6, 8, 12}

   Write the members of the union of Set H and Set J.

5. Set K is the set of odd numbers between 10 and 20.

   Set L is the set of even numbers between 10 and 20.

   Copy and finish: $K \cup L =$

6. Set R is the set of whole numbers between 47 and 48.

   Set S is the set of whole numbers larger than 15 and less than 17.

   Copy and finish: $R \cup S =$

7. X = {B, E} Y = {A, O, C, E}

   Copy and finish: $X \cup Y =$

8. $T = \{\frac{1}{2}, \frac{1}{5}, \frac{1}{4}, \frac{3}{4}\}$ $V = \{\frac{1}{3}, \frac{1}{5}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$

Set W is the union of Set T and Set V. Write the members of Set W.

9. $C = \{1, 6, 7, 8, 3\}$ $D = \{2, 6, 7, 8, 1\}$

Set E is the union of Set C and Set D. Write the members of Set E.

10. Set M is the set of whole numbers between 20 and 21. Set N is the set of even numbers between 6 and 8. Copy and finish: $M \cup N =$

11. $A = \{9, 18, 27, 36\}$ Set B has 5 members.

$A \cup B = \{3, 6, 9, 12, 15, 18, 27, 36\}$

$A \cap B = \{9\}$

$B =$

THE INTERSECTION OF SETS -- Exercise Set 11

1. A = {1, 2, 3, 4, 5}   B = {2, 4, 6, 8, 1}

   Which one of these sets is the intersection of Set A and Set B?

   M = {5, 6, 1, 2}
   N = {1, 2, 4}
   S = {1, 4, 6}

2. C = {a, e, i, o, u}   D = {a, b, c, d, e}

   Set E = the intersection of Set C and Set D. Write the members of Set E.

3. R = {5, 10, 15, 20, 25}   S = {10, 20, 30, 40}

   Set T is the intersection of Set R and Set S. Write the members of Set T.

4. Set X is the set of the first five counting numbers. Set Y is the set of odd numbers between 4 and 12. Set Z is the intersection of Set X and Set Y. Write the members of Set Z.

5. K = {dogs, cats, mice}   L = {pigs, dogs, cats, mice}

   M = {horses, cows, dogs}

   Copy and finish:

   $K \cap L =$
   $K \cap M =$
   $L \cap M =$

6. H = {s, t, u, d, y}   J = {h, a, r, d}

   Copy and finish: $H \cap J =$

7. A = {desk, chair, pencil}   B = {eraser, chalk, chair}

C = {book, tablet, desk}

Copy and finish:

$A \cap C =$

$A \cap B =$

$B \cap C =$

$C \cup A =$

$C \cup B =$

$B \cup A =$

8. J = {12, 18, 24, 30}. Set K has two members. One member is 6. $K \cap J = 18$. Copy and finish: K =

9. Set R is the set of counting numbers less than 6. Copy and finish: R =

Set S is the set of counting numbers between 5 and 11.

Copy and finish: S =

Copy and finish:

$R \cap S =$

$R \cup S =$

10. Set P is the set of all even numbers between 1 and 7.

Set Q is the set of all odd numbers between 1 and 7.

Copy and finish:

$P \cap Q =$

$P \cup Q =$

Chapter 2

NUMERATION

GROUPING IN BASE FIVE

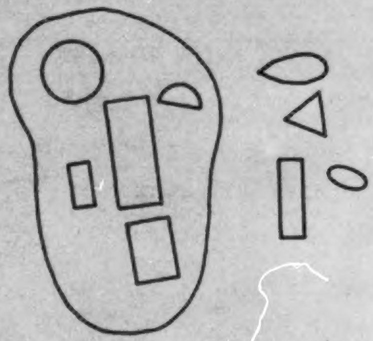
Our numeration system is sometimes called the base ten system. This means that we group in sets of ten. The base ten system uses only ten symbols. These are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We can write any numeral by using the ones of these ten symbols that we need.

This picture shows how the base ten system is used in writing numerals. On your paper write the letters a, b, c, d, e, and f. Beside each letter write the numeral which belongs at that place in the table.

	Group these objects into sets of ten.	How many sets of ten are there?	How many single objects are there remaining?	How do you write the base ten numeral?
1.	XXXXXXXXXX XXXXXX	1	6	16
2.	XXXXXXXXXX XXX XXXXXXXXXX	2	3	23
3.	XXXXXXXXXX XXXX	a	b	c
4.	XXXXXXXXXX XXXXX XXXXXXXXXX	d	e	f

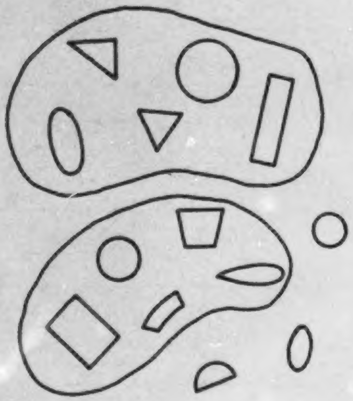
What would our numeration system look like if we tried to build a system which uses sets of five? Of course, in a base five system we would group by sets of five. Examples A, B, C, and D show how we can do this.

Example A:



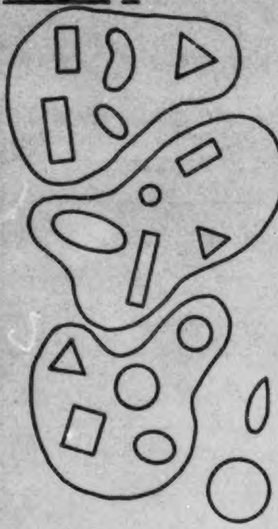
Let us put these objects into sets of five. We find one set of five and four single objects.

Example B:



In this picture, we have two sets of five objects each and three single objects.

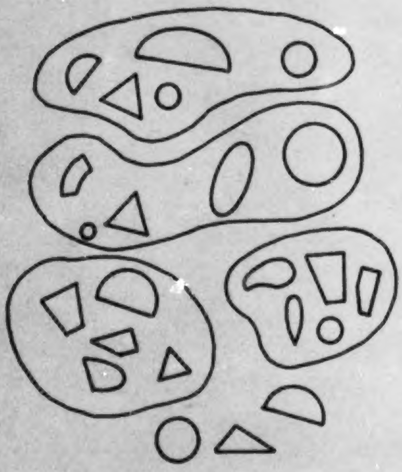
Example C:



Look at this set of objects.

1. How many sets of five are there?
2. How many objects remain?
3. We have __________ fives and __________ ones.

Example D:



1. How many sets of five are there?
2. How many objects remain?
3. We have __________ fives and __________ ones.

Exercise Set 1

1. Copy and finish this table.

Group these objects into sets of five.	How many sets of five are there?	How many single objects are there remaining?
///// ///// ///	2	3
ooooo oooo	___	___
xxxxx xxxxx xxxxx xx	___	___
□□□□□ □□□□□ □□ □□□□□ □□□□□	___	___

2. Draw a set of objects that will show 3 fives and 1 one.

3. Draw a set of objects that will show 2 fives and 3 ones.

4. Draw a set of objects that will show 4 fives and 2 ones.

5. Copy and complete this table.

Sets of Objects	Fives	Ones
xxxxx xxxx	1	4
ooooo ooooo ooooo oo	3	2
✓✓✓✓✓ ✓✓✓✓✓ ✓✓✓✓✓ ✓✓✓✓✓ ✓	___	___
□□□□□ □□□□□ □□□□	___	___
△△△△△ △△△△△ △△△△△ △△△△△	___	___

BASE FIVE NOTATION

Picture	Meaning	Numeral
x	1 one	1_{five}
xx	2 ones	2_{five}
xxx	3 ones	3_{five}
xxxx	4 ones	4_{five}
xxxxx	1 five and 0 ones	10_{five}
xxxxx x	1 five and 1 one	11_{five}
xxxxx xx	1 five and 2 ones	12_{five}
xxxxx xxx	1 five and 3 ones	13_{five}
xxxxx xxxx	1 five and 4 ones	14_{five}
xxxxx xxxxx	2 fives and 0 ones	20_{five}

When grouping in fives, we use place-value and symbols to name the number.

We need the symbols 0, 1, 2, 3, and 4 in order to write numerals in the base five system.

There is no one-place symbol in the base five system to mean one five. We regroup five ones as one five. We write this as the two-place numeral 10_{five}. This is read, "one zero, base five." The 0 is in the ones place to show there are no ones. The 1 is in the fives place to show one group of five.

Look at the six x's on the chart.
How many sets of five are there?
How many x's remain?
How is the base five numeral written?
How is this base five numeral read?

Suppose you have nine objects.
How many sets of five are there?
How many objects remain?
How would you write the base five numeral?
Read this base five numeral.

Think about thirteen objects.
How many sets of five are there?
How many objects remain?
How would you write the base five numeral?
Read this base five numeral.

Exercise Set 2

Each of these exercises has three parts. Write your answer for each part as is done for number 1.

1. Look at these objects.

   △ ○ ✓ ▽ ○ × □

   a) How many sets of five are there?

   b) How many objects remain?

   c) How is the base five numeral written? Answer: 1. a) 1
   b) 2
   c) 12_{five}

2. Look at these objects.

   × △ ○ z □ ☁ ▽ ▭ o ♡ ┌ ⊔

   a) How many sets of five are there?

   b) How many objects remain?

   c) How is the base five numeral written?

3. Look at these objects.

   □ △ ○ ⬭

   a) How many sets of five are there?

   b) How many objects remain?

   c) How is the base five numeral written?

4. Think about eight dogs.

   a) How many sets of five are there?

   b) How many of the eight dogs are not in the set of five?

   c) How is the base five numeral for eight written?

5. Think about nineteen cats.

   a) How many sets of five are there?

   b) How many of the nineteen cats are not in a set of five?

   c) How is the base five numeral for nineteen written?

Exercise Set 3

Copy and complete this chart. The first exercise is done for you.

Group these letters into sets of five.	How many sets of five letters are there?	How many letters remain?	How would you write the base five numeral?
G A D I B H E C F	1	4	14_{five}
K H A G B I C F E D J			
Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z			
G O F I P R C M K L T M N E Q S A B H			
A I C D H B L G F J K T N M R P O Q V S X U W			

Exercise Set 4

Copy and complete this chart. Number 1 shows what you are to do.

	Number of objects	Sets of five	Objects not in a set of five	Base five numeral
1.	11	2	1	21_{five}
2.	8			
3.	16			
4.	3			
5.	20			
6.	14			

Exercise Set 5

Count twenty objects. Answer the following questions.

1. Group the twenty objects into sets of nine.
    - a) How many sets of nine are there?
    - b) How many objects remain?
    - c) How many nines and how many ones equal twenty?
2. Group the twenty objects into sets of eight.
    - a) How many sets of eight are there?
    - b) How many objects remain?
    - c) How many eights and how many ones equal twenty?
3. Group the twenty objects into sets of seven.
    - a) How many sets of seven are there?
    - b) How many objects remain?
    - c) How many sevens and how many ones equal twenty?
4. Group the twenty objects into sets of six.
    - a) How many sets of six are there?
    - b) How many objects remain?
    - c) How many sixes and how many ones equal twenty?
5. Finish these. In exercise a), you should think of how many sets of nine there are in twenty objects. Then think how many objects are remaining.

a) 20 = _____nine c) 20 = _____seven

b) 20 = _____eight d) 20 = _____six

Exercise Set 6

Write the base five numeral for each of these exercises.

1. 9 = $\_\_\_\_\__{five}$
2. 14 = $\_\_\_\_\__{five}$
3. 17 = $\_\_\_\_\__{five}$
4. 15 = $\_\_\_\_\__{five}$
5. 3 = $\_\_\_\_\__{five}$
6. 22 = $\_\_\_\_\__{five}$

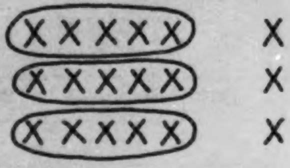
Write, in base ten, the number of objects which is meant by each of these base five numerals.

1. 23_{five} = _____ objects
2. 13_{five} = _____ objects
3. 2_{five} = _____ objects
4. 20_{five} = _____ objects
5. 33_{five} = _____ objects
6. 41_{five} = _____ objects

Exercise Set 7

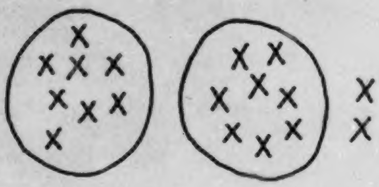
1. Here are some pictures of 18 x's. They are grouped in different ways. Tell how they are grouped. Then write the numeral. The first example is done for you.

a.

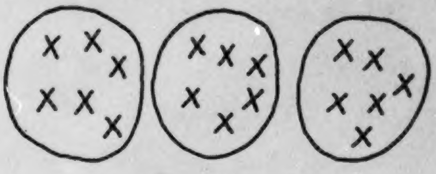


3 fives and 3 ones = 33_{five}.

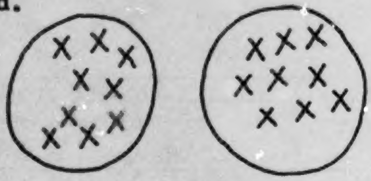
b.



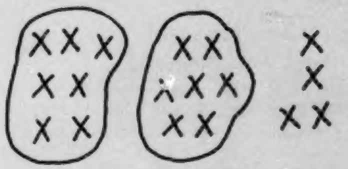
c.



d.



e.



2. Group 32 x's into the following sets. Each time think how many sets there are and how many remain. Then write the number.

a) Sets of eight

b) Sets of seven

c) Sets of nine

d) Sets of six

After you have completed Exercise Set 7 you will see that Exercise Set 8 is very much like Set 7. Here we want you to think of the number of x's and how to group them. But do not make the x's on your paper--just think of them.

Let us take the next to the last row of the table in Exercise Set 8 and see how we can find what numerals to write in the empty spaces.

Base Ten Numeral	Sevens	Ones	Base Seven Numeral
20			

The base ten numeral is 20. Think of 20 x's made on your paper--do not make them, just think of them. Next think of how many sets of seven there are in 20. Are there just 2 sets of seven each in 20? Yes. Also there are 6 ones left over. How can you make a record of this in the table? You can put a 2 below Sevens and a 6 under Ones. Then below Base Seven Numeral you can write the numeral which means 2 sevens and 6 ones. This numeral is 26_{seven}. Then the next to the last row of the table will look like this:

Base Ten Numeral	Sevens	Ones	Base Seven Numeral
20	2	6	26_{seven}

Now complete the table in Exercise Set 8.

Exercise Set 8

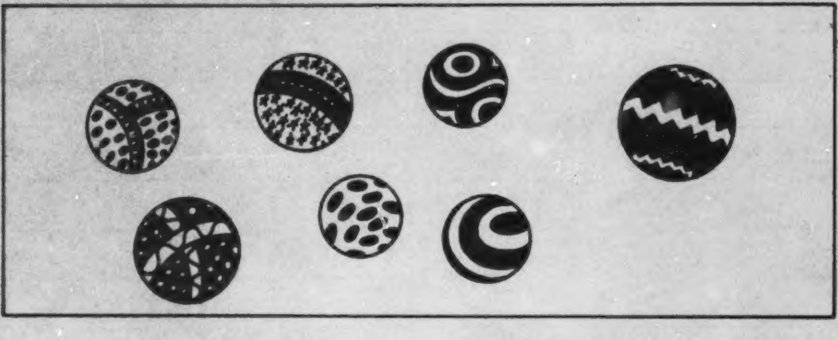
Complete this table.

Base Ten Numeral	Threes	Ones	Base Three Numeral
5			
3			
8			
Base Ten Numeral	Sixes	Ones	Base Six Numeral
11			
22			
18			
Base Ten Numeral	Fours	Ones	Base Four Numeral
9			
14			
11			
Base Ten Numeral	Eights	Ones	Base Eight Numeral
25			
16			
19			
Base Ten Numeral	Sevens	Ones	Base Seven Numeral
13			
20			
10			

A NUMBER MAY HAVE SEVERAL NAMES

Here is a set of balls.

The number of members in the set is seven.



What can we write on the chalkboard to name the number of balls?

Is each of these numerals a name for seven: 13_{four}, 21_{three}, 12_{five}, 11_{six}? Yes, they are. Some of these numerals are the kind of names you have been studying in this unit. They are all good names. Also these names are sometimes used for seven: VII and 𝍸 ||

There are still other names for seven. Is each one of these a name for seven: $8 - 1$, $3 + 4$, $10 - 3$, $5 + 2$? Yes. There are many more ways that we may name 7. What are some of the names?

Exercise Set 9

Write different names for the numbers. Use only base ten numerals. Example: 56 = 50 + 6; 56 = 40 + 10 + 6; 56 = 60 - 4; 56 = 55 + 1.

1. Write four different names for 8.

2. Write four different names for 27.

3. Write four different names for 64.

4. Write four different names for 1.

5. Write four different names for 163.

6. Write four different names for 378.

7. Write four different names for 500.

RENAMING NUMBERS

Example A: The number 23 may be renamed as

2 tens and 3 ones.

1 ten and 13 ones.

Example B: Let us see how many ways we can think of for naming the number 457. In renaming use only ones, tens, and hundreds.

457 = 457 ones

457 = 4 hundreds + 5 tens + 7 ones

457 = 4 hundreds + 4 tens + 17 ones

457 = 3 hundreds + 15 tens + 7 ones

457 = 3 hundreds + 14 tens + 17 ones

457 = 45 tens + 7 ones

457 = 400 + 50 + 7

Example C: The number 4,605 can be renamed in many ways. Ones, tens, hundreds, and thousands can be used in renaming.

4,605 = 4,605 ones

4,605 = 46 hundreds + 0 tens + 5 ones

4,605 = 4 thousands + 6 hundreds + 5 ones

4,605 = 460 tens + 5 ones

4,605 = 4,000 + 600 + 5

4,605 = 4,600 + 5

Exercise Set 10

In exercises 1 through 10, write T if the sentence is true. Write F if the sentence is false.

1. 34 may be renamed as 3 tens + 4 ones.
2. 34 may be renamed as 2 tens + 14 ones.
3. 34 may be renamed as 4 tens + 3 ones.
4. 34 may be renamed as 1 ten + 24 ones.
5. 365 may be renamed as 3 hundreds + 5 tens + 15 ones.
6. 365 may be renamed as 35 tens + 5 ones.
7. 365 may be renamed as 2 hundreds + 15 tens + 15 ones.
8. 365 may be renamed as 3 hundreds + 60 tens + 5 ones.
9. 184 may be renamed as 18 tens + 14 ones.
10. 184 may be renamed as 1 hundred + 7 tens + 14 ones.

For each exercise from 11 through 16 write the base ten numeral.

11. Six hundreds + five tens + three ones
12. Three hundreds + twelve tens + eight ones
13. Nine hundreds + four tens + fifteen ones
14. 4 hundreds + 17 tens + 8 ones
15. 7 hundreds + 5 tens + 16 ones
16. 8 thousands + 6 hundreds + 3 tens + 12 ones

Exercise Set 11

Name each of the following in two ways as a number of tens and a number of ones. Number 1 is answered for you.

1. 37 **Answer:**
3 tens and 7 ones
2 tens and 17 ones

2. 26

3. 54

4. 80

5. 77

6. 39

Complete the following sentences.

7. 354 = 3 hundreds + _____ tens + 4 ones.

8. 354 = 3 hundreds + 4 tens + _____ ones.

9. 354 = 2 hundreds + _____ tens + 14 ones.

10. Name 836 in three different ways as hundreds, tens, and ones.

11. Name 605 in three different ways as hundreds, tens, and ones.

12. Name the number 612 in four different ways.

13. Name 7,631 in three different ways as thousands, hundreds, tens, and ones.

14. Name the number 3,806 in four different ways.

15. Name each of the following as a base ten numeral.

a) Four hundreds + five tens + six ones = ________

b) Two thousands + three hundreds + seven tens + five ones = ________

c) Six hundreds + twelve tens + thirteen ones = ________

d) Nine hundreds + thirteen tens + fourteen ones = ________

e) 12 hundreds + 14 tens + 19 ones = ________

Exercise Set 12

1. From the list below write all the letters which are beside correct names for 467.

   a) Four hundred sixty-seven
   b) Forty-six and seven more
   c) Forty-six tens and seven
   d) Forty hundreds + sixty-seven ones
   e) 300 + 160 + 7
   f) Seven plus four hundred
   g) 400 + 60 + 7
   h) 300 + 150 + 17
   i) 467 tens

2. Answer Yes or No.

   a) 3,729 is 37 tens plus 29 ones.
   b) Ten hundreds plus forty tens plus nine ones is the same as one thousand forty-nine.
   c) 5,000 + 500 + 1 = 5,501
   d) 36 hundreds + 1 ten + 18 ones = 3,628
   e) 734 = 600 + 120 + 24

3. Write the base ten numeral for each.

   a) Five thousands + six hundreds + eight tens + three ones
   b) 3 thousands + 8 hundreds + 16 tens + 5 ones
   c) 6 thousands + 15 hundreds + 2 tens + 7 ones
   d) 8 thousands + 4 hundreds + 14 tens + 16 ones
   e) 9 thousands + 12 hundreds + 3 tens + 14 ones

EXTENDING IDEAS OF THE DECIMAL SYSTEM

We know that each place in a numeral written in the decimal system has a name. In a whole number the first place on the right is the ones' place, the second the tens' place, and the third the hundreds' place. The fourth position is the thousands' place, the fifth the ten thousands' place, and the sixth is the hundred thousands' place.

We also know that:

10 ones are the same as 1 ten;
10 tens are the same as 1 hundred;
10 hundreds are the same as 1 thousand;
10 thousands are the same as 1 ten thousand;
10 ten thousands are the same as 1 hundred thousand.

Place Name	Hundred thousand	Ten thousand	One thousand	Hundred	Ten	One
Digits	2	2	2,	2	2	2

In the numeral shown on the chart, each 2 has a different position. Tell the place value of each 2.

We now want to read the names of large numbers. These will be numerals with as many as six digits but with no more than six.

You can read large numerals of as many as six digits easily if the three digits at the right are separated from the others by a comma, as in 222,222 or in 74,609. The number named by the digits to the left of the comma tell how many thousands there are; the number named by the digits to the right of the comma tells how many ones there are.

We read 222,222 as "two hundred twenty-two thousand, two hundred twenty-two." We read 74,609 as "seventy-four thousand, six hundred nine." Notice that the word "and" is not used in any way in reading numerals like these.

Read these numerals.

736,421
80,592
603,250
248,759
10,003
5,791
890,602
927,030

Exercise Set 13

Write the letter which is in front of the correct way to say each of these numerals. Exercise 1 is done for you.

1. 28 a) twenty-seven Answer: 1. c
b) eighty-two
c) twenty-eight
d) eight

2. 5,250 a) five thousand, two hundred five
b) five hundred twenty-five
c) five thousand, one hundred fifty
d) five thousand, two hundred fifty

3. 17,002 a) seventeen thousand, two hundred
b) seventeen thousand, two
c) seventeen thousand, twenty
d) one hundred seventy thousand, two

4. 156,946 a) one hundred fifty-six thousand, nine hundred forty-six
b) one hundred sixty-five thousand, nine hundred forty-six
c) one hundred fifty-six million, nine hundred forty-six
d) one fifty-six thousand, six hundred forty-nine

5. 108,200
a) one hundred eighty million, two hundred thousand
b) one hundred eight thousand, two hundred
c) one hundred thousand, eight hundred twenty
d) one hundred eight million, two

6. 100,007
a) one hundred thousand, seven hundred
b) one hundred thousand, seventy
c) one hundred thousand, seven
d) one thousand, seven

7. 365,843
a) three hundred fifty-six thousand, eight hundred forty-three
b) three hundred sixty-five thousand, eight hundred forty-three
c) three hundred sixty-five thousand, eight hundred thirty-four
d) three hundred sixty-five million, eight hundred forty-three

8. 22,222
a) twenty-two thousand, two hundred twenty-two
b) two hundred twenty-two thousand, twenty-two
c) two hundred twenty-two million, two
d) two hundred two thousand, two

Write numerals to represent each of the following.

9. Three hundred six

10. Six thousand, seven hundred fifty-six

11. Forty-seven thousand, six hundred four

12. Forty thousand, one hundred twenty-five

13. Three hundred fifty-one thousand, five hundred sixty-four

14. Forty thousand, forty

15. Two hundred fifty thousand, fifty-six

16. Seven hundred fifty-six thousand, thirty

17. Five hundred thousand, five

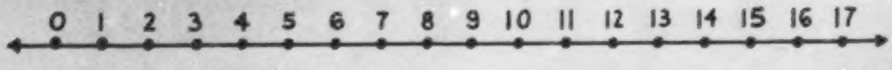
18. Four hundred four thousand, four

19. Six thousand, six

20. Sixty thousand, six

21. Sixty thousand, six hundred six

ORDER RELATIONS ON THE NUMBER LINE



We have talked about the number line. We know that it is a straight line with points on it which have been matched with the set of whole numbers. The point zero is at the left. If you think of two numbers represented on a number line, the one to the right is larger than the one to the left.

In drawing a number line, do we have to show the point that is labeled zero? Of course we do not. It is possible to show just a part of the number line. If we start with the smallest whole number, then we start with zero. But if we label the first point 132, then the next points to the right, in order, will be 133, 134, 135, and so on.

Exercise Set 14

1. Draw a number line showing the whole numbers from 256 through 270.

In each of the following exercises, find the points representing the two numbers on the number line. Write the missing numbers on your paper.

2. 266 and 258

________ is less than ________

________ is to the left of ________

________ is to the right of ________

3. 261 and 269

_______ is less than _______

_______ is to the right of _______

_______ is to the left of _______

4. 270 and 263

_______ is greater than _______

_______ is to the left of _______

_______ is to the right of _______

5. 264 and 268

_______ is less than _______

_______ is to the left of _______

_______ is to the right of _______

6. 257 and 260

_______ is greater than _______

_______ is to the right of _______

_______ is to the left of _______

SOME NEW SYMBOLS

We usually write "2 + 5 equals 7" using the equals sign (=) instead of writing out the word "equals". We write:

$$2 + 5 = 7.$$

We have another symbol which is called the "less than" sign. To write "2 is less than 3," we write

$$2 < 3.$$

We also have a "greater than" sign. To write "8 is greater than 3," we write

$$8 > 3.$$

Complete the following statements with "less than" or "greater than".

1. $5 < 7$ Five is ________ ________ seven.

2. $12 > 9$ Twelve is ________ ________ nine.

3. $0 < 1$ Zero is ________ ________ one.

4. $6 > 0$ Six is ________ ________ zero.

5. $201 > 198$ Two hundred one is ________ ________ one hundred ninety-eight.

6. $5 < 327$ Five is ________ ________ three hundred twenty-seven.

Exercise Set 15

Write each of these sentences in the short way.

EXAMPLE: Seven is less than ten. $7 < 10$

1. One is greater than zero.

2. Eleven is less than thirteen.

3. Fifty-six is greater than twenty-one.

4. Two hundred sixty is less than three hundred sixty.

5. Three hundred fifty-nine is greater than two hundred ninety-seven.

6. Two hundred sixty-two is less than three hundred.

Write the numerals 1 to 12 in a column on your paper. Write T if the statement is true and F if the statement is false.

1. $4 < 7$
2. $5 > 1$
3. $4 > 4$
4. $0 < 5$
5. $30 < 17$
6. $52 > 49$
7. $120 < 19$
8. $299 > 617$
9. $426 > 425$
10. $629 < 89$
11. $513 > 377$
12. $201 < 210$

Exercise Set 16

Copy each mathematical sentence below. Fill in the "less than", "greater than", or "equals" sign in order to make a true statement.

1. (8 + 4) _____ 11
2. (1 + 3) _____ (9 + 1)
3. (2 × 3) _____ (3 × 3)
4. (2 + 7) _____ (11 - 2)
5. (2 + 3) + 4 _____ 1 + (5 + 3)
6. (3 × 4) _____ (2 × 5)
7. (12 + 50) _____ 59
8. (20 + 15) _____ (10 + 25)
9. (8 + 3) + 2 _____ (7 + 5) + 2
10. (24 - 2) _____ (3 × 7)
11. (18 - 6) _____ (9 + 3)
12. (5 + 8) - 3 _____ (14 - 7) + 5

JUST FOR FUN

Make a copy of this puzzle and fill it in.

A	B		▨	▨	C	D	E
F		▨	G	H	▨	I	
	▨	J			K	▨	
▨	L		▨	▨	M	N	▨
▨	O		▨	▨	P		▨
Q	▨	R	S	T		▨	U
V		▨	W		▨	X	

Across

A. Two hundred seventy-three
C. Another name for 400 + 30 + 9
F. 1 more than 4 tens + 8 ones
G. 3 eights and 2
I. (60 + 3) + (3 + 3) = ?
J. 46 hundreds + 17 ones
L. _?_ is 3 groups of seven
M. 4 nickels and 7 cents
O. (3 × 7) + 9
P. _?_ is 12 more than 70
R. 3000 + 500 + 60 + 9
V. 539 = _?_ tens + 19 ones
W. How many inches equal one foot?
X. XVII is another name for _?_

Down

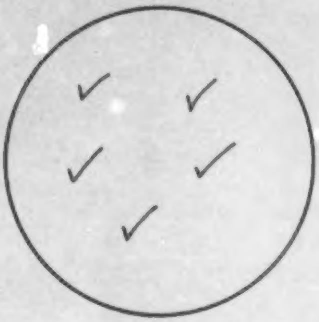
A. 24 tens
B. _?_ < 80
D. 3 dozen
E. One less than one thousand
G. 2 tens and 6 ones
H. _?_ > 52
J. Four thousand, one hundred three
K. 4,289; 5,289; 6,289; _?_
L. 1 ten and 13 ones
N. 720 equals how many tens?
Q. 3 fours + 3 ones
S. 5,264 = _?_ hundreds + 15 tens + 14 ones
T. Five tens and twelve ones
U. 7, 12, 17, 22, _?_

Chapter 3

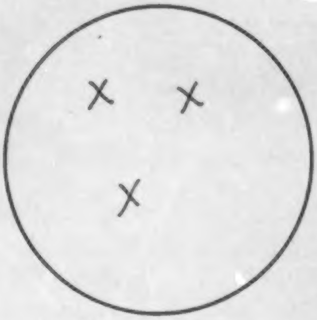
PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION I

ADDITION AND SUBTRACTION

1.



Set A



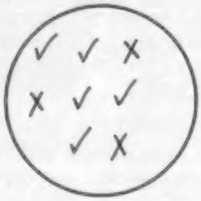
Set B

(a) How many members are in Set A?

(b) How many members are in Set B?

(c) How many members are in the union of Set A and Set B?

A picture of $A \cup B$ is:



$A \cup B$

When we find the number of members in the union of two sets which have no common members we may use addition.

2. Set G is a set of nine year old girls in a fourth grade room.

G = {Mary, Betty, Jean, Helen}

Set B is a set of girls in this same room who have brown hair.

B = {Linda, Betty, Helen, Sandra, Nancy}

(a) What is the number of members of Set G?

(b) What is the number of members of Set B?

(c) What is the number of members of $G \cup B$?

We could not add 4 and 5 to find the number of members of the union of these two sets.

We can use addition to find the number of elements in the union of two sets only if the two sets have no common members.

Exercise Set 1

Answer each question carefully.

1. Set F = {dog, cow, horse, pig, turkey}

   Set G = {chicken, dog, robin, cat, pig}

   (a) The number of members of Set F is ?

   (b) The number of members of Set G is ?

   (c) The number of members of $F \cup G$ is ?

   Why couldn't we add the number of members of Set F and the number of members of Set G to get the number of members of $F \cup G$?

2. J = {a, b, c, d, e, f, g, h, i}

   K = {j, k, l, m}

   (a) What is the number of members of $J \cup K$?

   (b) Could you add the number of members of Set J and the number of members of Set K to find the answer? Why?

3. M = {f, o, u, r, t, h}

   P = {g, r, a, d, e}

   What is the number of members of $M \cup P$?

4. There are 8 members in Set R.

   There are 5 members in Set S.

   No members of Set R are members of Set S.

   What is the number of members of $R \cup S$?

   How did you find your answer?

5. Set E has 9 members.

Set F has 7 members, none of which are members of Set E.

What is the number of members in $E \cup F$?

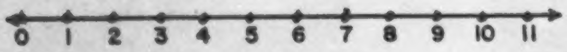
6. $W = \{5, 7, 9, 10, 11, 12, 13, 14\}$

$Y = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22\}$

The number of members in $W \cup Y$ is ?

How did you find your answer?

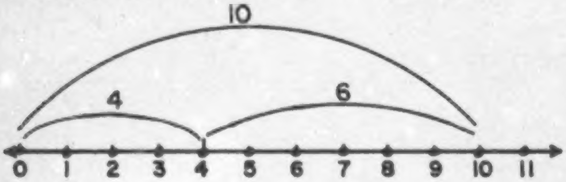
ADDITION AND THE NUMBER LINE



A part of the number line between two points is called a line segment. For example the part of the line between the point labelled 3 and the point labelled 6 is called "the line segment from 3 to 6". The points labelled 3 and 6 belong to this line segment. Name some other line segments by looking at the number line above.

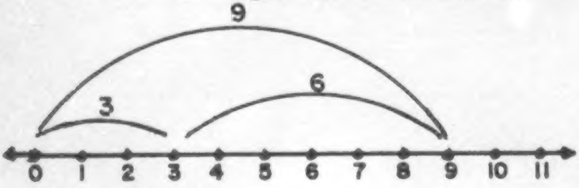
We can use the number line and line segments to help us picture the addition of numbers.

This picture helps us "see" that 4 + 6 = 10

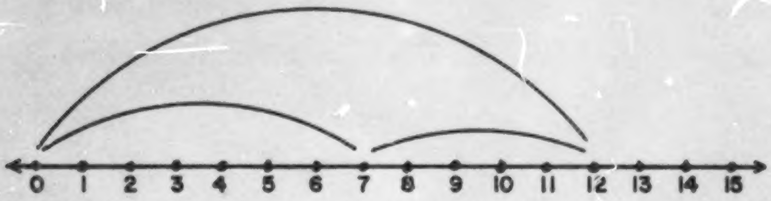


We first draw the curve to picture 4. Next, we draw the curve to picture 6. We start this at 4 and go 6 units to the right. Then we draw the curve from 0 to 10 to picture the sum of 6 and 4.

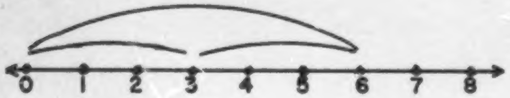
A picture of 3 + 6 = 9 might look like this:

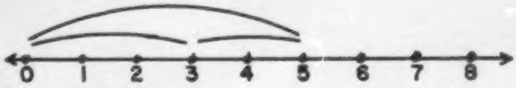


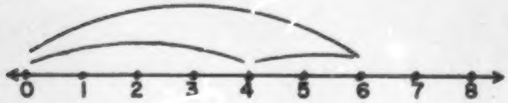
1. What does this picture suggest to you?


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

2. Which of these is a picture of 3 + 2 = 5?

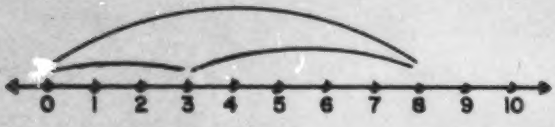
(a)


(b)


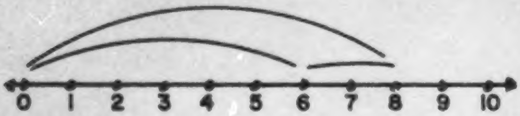
(c)


Exercise Set 2

1. Is this a picture of 4 + 3 = 7 or of 3 + 5 = 8?


0
1
2
3
4
5
6
7
8
9
10

2. What does this picture suggest to you?


0
1
2
3
4
5
6
7
8
9
10

3. Use a number line to draw a picture of 2 + 5 = 7.

4. Draw a number line picture of each of these.

(a) 7 + 6 = 13

(b) 5 + 9 = 14

(c) 4 + 7 = 11

ADDITION AND SUBTRACTION AS OPERATIONS

Addition and subtraction are two operations of mathematics. Multiplication and division are also operations. These four operations are called the basic operations.

An operation on two numbers is a way of thinking about two numbers and getting one and only one number. When we think about 9, 5 and get 14, we are adding. We write 9 + 5 = 14. When we think about 9, 5 and get 4, we are subtracting. We write 9 - 5 = 4. In subtracting, the order of the two numbers is important. If the numbers are 5, 9 this would mean 5 - 9. There is no whole number for 5 - 9.

Exercise Set 3

Number from 1 through 12 on your paper. Write the correct numeral or word to complete this chart. The first exercise is done for you.

	Numbers Operated On	Result	Operation Used
1.	5, 7	12	Addition
2.	9, 3	___	Subtraction
3.	10, 2	12	___
4.	10, 2	8	___
5.	10, 2	5	___
6.	10, 2	20	___
7.	5, ___	9	Addition
8.	___, 9	7	Subtraction
9.	9, ___	3	Subtraction
10.	6, 9	15	___
11.	___, 7	4	Subtraction
12.	3, ___	1	Subtraction

TRUE MATHEMATICAL SENTENCES

A sentence which tells us something about numbers is a mathematical sentence.

Statements like those in Box A are called mathematical sentences. A mathematical sentence can be true or false.

A

$6 + 4 = 10$

$9 - 3 = 2$

$0 + 8 = 8$

1. (a) Is it true that $6 + 4 = 10$?
   (b) Is it true that $9 - 3 = 2$?
   (c) Is it true that $7 + 6 \neq 12$?
   (d) Is it true that $10 - 10 \neq 10 - 9$?
2. (a) Are $6 + 4$ and 10 different names for the same number?
   (b) Are these names for the same number: $6 + 3$, $11 - 2$, $9 + 0$?
   (c) The base ten numeral for $9 - 3$ is 6. What is the base ten numeral for: $17 - 3$; $12 + 5$; $9 - 0$?
3. Write some other mathematical sentences.
4. Mary said, "I'm thinking of a number. The number is the result of adding 5 and 3. Of what number am I thinking?"
5. Bob said, "Let's call Mary's number n. Then, n is the result of adding 5 and 3. So, $n = 8$. Now, if $n = 8$, then $5 + 3 = n$."

   Statements like those in Box B are also called mathematical sentences.

B

$3 + 8 = n$

$12 - 7 = n$

$n - 4 = 7$

$4 - n = 1$

6. (a) If $n = 5$, is $2 + 3 = n$?

(b) If $n = 9$, is $n - 5 = 4$?

(c) If $n = 5$, is $n - 5 = 1$?

(d) If $n = 7$, is $3 + 5 = n$?

7. (a) If $n = 4$, is $10 - n = 6$?

(b) If $n = 7$, is $n + 3 = 10$?

(c) If $n = 9$, is $n + 3 = 10$?

(d) If $n = 7$, is $10 - n = 2$?

8. (a) If $n = 8 + 7$, is $9 + 6 = n$?

(b) If $n = 30 - 20$, is $15 + n = 25$?

(c) If $n = 100 + 300$, is $n - 200 = 200$?

9. What is the base ten numeral for n in each of (a), (b), (c), in Exercise 8.

10. (a) If $n = 7$, is $11 - 3 \neq n$ or is $11 - 3 = n$?

(b) If $n = 12$, is $17 - 5 \neq n$ or is $17 - 5 = n$?

(c) If $n = 1$, is $12 - n \neq 11$ or is $12 - n = 11$?

11. If $n = 7$, which of these are different names for n?

(a) $12 - 5$ (b) $19 - 13$ (c) $10 - 7$ (d) $27 - 20$

12. Joan said, "6 + 5 = 11 is a mathematical sentence. It is true." Do you think Joan is right? Draw a model or picture to show why you think as you do.

13. Bill said, "6 - 2 = 3 is a mathematical sentence. It is true." Do you agree with Bill? Draw a picture or model to show why you think as you do.

The mathematical sentence 6 + 7 = 13 is true. It is true because 6 + 7 and 13 are different names for the same number.

The mathematical sentence 6 + 7 = 12 is not true. It is not true because 6 + 7 and 12 are not different names for the same number.

Exercise Set 4

1. Some of these mathematical sentences are true. Write on your paper the letter of each mathematical sentence that is true.

(a) 2 + 1 = 3
(b) 6 + 8 = 15
(c) 9 + 14 = 23
(d) 11 + 1 = 11
(e) 13 - 8 = 4
(f) 9 - 3 = 6
(g) 17 - 11 = 6
(h) 19 - 12 = 6
(i) 26 + 21 = 74
(j) 33 - 21 = 2
(k) 62 + 6 = 122
(l) 78 - 62 = 16

2. If $n = 5$, which of these mathematical sentences are true?

(a) $1 + 4 = n$ (b) $8 - n = 3$ (c) $n - 2 = 3$

3. Suppose you were asked, "What number must n represent so that $8 + 5 = n$ is a true mathematical sentence?"

(a) If you said, "n is 13," would you be correct?
(b) If you said, "n is 12," would you be correct?
(c) Would it be correct to say, "If $n = 13$, then $8 + 5 = n$?"

4. If n represents the number 3, which of these sentences are true?

(a) $2 + n = 5$ (d) $9 - n = 5$
(b) $8 + n = 10$ (e) $6 - n = 3$
(c) $n + 3 = 6$ (f) $3 - n = 0$

5. If $n = 7$, which of these are true mathematical sentences?

(a) $n + 9 = 16$ (c) $9 - n = 2$ (e) $8 + n = 12$
(b) $n - 6 = 1$ (d) $4 + 3 = n$ (f) $12 - n = 8$

6. (a) What number is represented by n so that $8 + 4 = n$ is a true mathematical sentence?
(b) What number is represented by n so that $2 + n = 11$ is a true mathematical sentence?

7. What number is represented by n so that $3 + n = 9$ is a true sentence? You may use the form in Box A to write your answer.

A
$3 + n = 9$
$n =$

8. What number is n so that $2 + n = 7$ is a true sentence? Write your answer in the same form that you used in exercise 7.

Exercise Set 5

Copy the numerals 1-15 on your paper. Next to each, write the correct words, numerals and mathematical sentences to complete this chart. The first one is done for you.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
1.	12, 9	3	Subtraction	$12 - 9 = 3$
2.	18, 9	9	________	$18 - 9 = 9$
3.	6, 3	9	________	$6 + 3 = 9$
4.	5, 8	40	________	$5 \times 8 = 40$
5.	18, 3	__	________	$18 \div 3 = 6$
6.	6, m	13	Addition	________
7.	p, 7	6	Subtraction	________
8.	3, 7	__	Addition	________
9.	14, t	8	Subtraction	________
10.	12, __	5	Subtraction	________
11.	15, 9	6	________	________
12.	3, 14	17	Addition	________
13.	12, 4	__	Division	________
14.	5, 3	__	Multiplication	________
15.	6, 2	__	Subtraction	________

THINKING ABOUT ADDITION FACTS

You show addition like this:

9 + 5 = 14 $$\begin{array}{r} 9 \\ +5 \\ \hline 14 \end{array}$$

9	+	5	=	14
addend		addend		sum

You read addition like this:

9 and 5 are 14, or 5 added to 9 is 14, or 9 plus 5 is 14, or 9 plus 5 equals 14, or 9 + 5 is equal to 14.

In each of these examples tell which numbers are addends and which is the sum.

(a) 8 + 5 = 13

(b) 8 + 9 = 17

(c) 10 + 20 = 30

(d) $$\begin{array}{r} 31 \\ +45 \\ \hline 76 \end{array}$$

(e) $$\begin{array}{r} 23 \\ +64 \\ \hline 87 \end{array}$$

You may know all the addition facts. This will help you in completing the chart on the next page. It is called an addition chart. To begin, you add to the number 0, which is in the left column, each of the numbers in the top row. Write the sum under the number which was added to zero. Add, 0 + 0 = 0 and write 0 under zero; 0 + 1 = 1, so write 1 under 1, and so on. Make a copy of the chart on the next page and finish it.

Addition Chart

+	0	1	2	3	4	5	6	7	8	9
0	0	1								
1										
2										
3										
4										
5										
6										
7										
8										
9										

Exercise Set 6

A

346 901

+198 +648

Addition of numbers as in the box may be done quickly if you know the addition facts. You should be able to recall all addition facts given in the addition chart. Here are a few ideas in case you forgot some facts.

1. Complete these statements:

   (a) Because 7 + 7 = 14, 7 + 8 = _____

   (b) Because 6 + 6 = 12, 6 + 7 = _____

   (c) Because 5 + 5 = 10, 6 + 5 = _____

   (d) Because 8 + 8 = 16, 8 + 7 = _____

2. (a) You know 6 + 8 = 14. How do you find 6 + 9?

   (b) You know 9 + 4 = 13. How do you find 9 + 5?

   (c) You know 7 + 9 = 16. How do you find 8 + 9?

3. Complete these statements:

   (a) Because 10 + 9 = 19, 9 + 9 = _____.

   (b) Because 10 + 8 = 18, 9 + 8 = _____.

   (c) Because 7 + 10 = 17, 7 + 9 = _____.

   (d) State a way to add 9 to any number.

4. Complete these statements:

(a) 0 + 5 = ____

(b) 9 + 0 = ____

(c) 6 + 0 = ____

If you add 0 to any number, what is the sum?

5. Complete these statements:

(a) 9 + 1 = ____

(b) 7 + 1 = ____

(c) 1 + 8 = ____

If you add 1 to any number, what is the sum?

6. Look at the addition chart you made. Name the numbers in each of the sets described below.

(a) The members of set A are found by adding 9 to each of the numbers in the set
{0, 1, 2, 3, ..., 9}

(b) The members of set B are obtained by adding 7 to each of the numbers in the set
{0, 1, 2, 3, ... 9}

(c) The members of set C are obtained by adding 6 to each of the numbers in the set
{0, 1, 2, 3, ..., 9}

7. **BRAINTWISTER:** Use your answer to exercise 6 to find:

(a) $A \cap B$	(d) $A \cup B$
(b) $A \cap C$	(e) $A \cup C$
(c) $B \cap C$	(f) $B \cup C$

8. BRAINTWISTER:

(a) Use your answer to exercise 6 to find

$(A \cap B) \cap (B \cap C)$.

(b) Your answer for Exercise (a) is the same as which answer of Exercise 7?

THE COMMUTATIVE PROPERTY FOR ADDITION

1. Pretend you have a dime and a quarter. You are paying for a 35 ¢ book with this quarter and dime. Would the order of giving the coins to the clerk make an difference in what you paid him?

2. Use the letters "O" and "N" to make two words. What are the words? Did the order of the letters change the result?

3. Is the result the same for each addition in (a)? in (b) ? in (c)?

(a)	(b)	(c)
7 + 6, 6 + 7	50 + 47, 47 + 50	0 + 891, 891 + 0

How are the sums different in (a)? in (b)? in (c)?

4. (a) Is 400 + 500 = 500 + 400?
   (b) Is 692 + 8 = 8 + 692?
   (c) Is 1,000,000 + 0 = 0 + 1,000,000?
   (d) If n is a whole number, is n + 10 = 10 + n?

ADDITION IS A COMMUTATIVE OPERATION

For example, 3 + 5 = 8
5 + 3 = 8

The sum is the same even if the order of the addends is changed. So we can write

3 + 5 = 5 + 3.

Exercise Set 7

1. Is $5 - 2 = 2 - 5$? Is $9 - 7 = 7 - 9$? Do you know what number $2 - 5$ is? Does the commutative property seem to hold for subtraction?

2. A teacher was reading to a class, "What number is represented by n so that $637 + 596 = n$?" Jim did not hear the 637 so he wrote, $596 + \underline{\hspace{3em}} = n$. He asked the teacher to tell him the first addend. He then wrote, $596 + 637 = n$. Will his result be the same as the pupil who wrote $637 + 596 = n$? Why?

3. (a) Is $13 - 11 = 11 - 13$?

   (b) Is $20 - 10 = 10 - 20$?

   (c) If $n = 10$, is $n + 6 = 6 + n$?

   (d) If $n = 20$, is $n + 6 = 6 + n$?

   Tell what number n represents so that $n - 6 = 6 - n$.

4. Which of the following are true mathematical sentences?

   (a) $18 + 11 = 11 + 18$

   (b) $203 + 401 = 200 + 404$

   (c) $6 + 5 = 7 + 4$

   (d) $1{,}207 + 2{,}011 = 1{,}102 + 7{,}021$

   (e) $19 + 91 = 91 + 19$

   (f) $95 + 59 = 59 + 59$

5. Which mathematical sentences in exercise 4 illustrate the commutative property?

6. BRAINTWISTER: (a) Write the numerals for whole numbers (if possible) to complete this chart.

Numbers Operated On	Result	Operation Used
15, 7	____	Addition
7, 15	____	Addition
15, 7	____	Subtraction

(b) Is there a whole number for each blank? If not, why?

7. BRAINTWISTER: (a) What whole number is represented by n so that $0 + n = n + 0$?

(b) What whole number is represented by n so that $12 - n = n - 12$?

(c) What whole numbers are represented by x and y so that $x + y = y + x$?

THINKING ABOUT SUBTRACTION FACTS

Subtraction is the operation of finding the unknown addend if we know the sum and one addend. For example, if $8 + n = 12$ then 8 is one addend and n is the unknown addend. We subtract 8 from 12 to find the number that n represents.

You show subtraction like this:

$9 - 5 = 4$

$$\begin{array}{r}9\\ -5\\ \hline 4\end{array}$$

The names of the parts in a subtraction sentence are:

9	-	5	=	4
sum		addend		addend

You read subtraction like this: 9 minus 5 equals 4 or 5 subtracted from 9 is 4.

In each of these examples tell which numbers are addends and which number is the sum.

(a) $15 - 9 = 6$
(c) $n + 5 = 13$
(e) $17 - n = 8$
(b) $9 = 13 - 4$
(d) $3 + n = 12$
(f) $14 - 8 = n$

You know many subtraction facts. We will call the chart in which we list them a subtraction chart. Copy the chart on the next page and fill it in.

You begin with zero in the top row. Subtract it from each number in the first column. For example, $0-0 = 0$; place the result, 0, to the right of the zero that is in the first column, $1 - 0 = 1$; write 1 to the right of the 1, and so on. Go on. To get your answers, think like this for $8 - 5$: "What added to 5 is 8?" Each of the numbers in the first column is a sum. All the other numbers in the chart are addends.

Subtraction Chart

-	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Exercise Set 8

Use your subtraction chart to help you answer these questions.

1. Why is the first column which you wrote the same as the column to its left?

2. Why do the numbers in each row decrease to zero?

3. Why do the numbers in each column increase?

4. Why is part of the chart empty?

5. Study the subtraction chart you finished. Write these sets of numbers:

   (a) The members of set X are the only possible numbers that can be addends if the sum is 3.

   (b) The members of set Y are the only possible numbers that can be addends if the sum is 7.

   (c) The members of set Z are the only possible numbers that can be addends if the sum is 9.

6. **BRAINTWISTER:** Use your answer to Exercise 5 to find:

   (a) $X \cap Y$ (c) $Y \cap Z$ (e) $X \cup Z$
   (b) $X \cap Z$ (d) $X \cup Y$ (f) $Y \cup Z$

7. **BRAINTWISTER:** Use Your answer to Exercise 6 to find $(X \cap Y) \cap (Y \cap Z)$.

Exercise Set 9

A

892   310
-375   -184

Subtraction of numbers as in Box A may be done quickly and accurately if you know the addition facts. You should be able to recall all unknown addends in the addition chart.

1. Tell how to locate the unknown addend for 9 - 4 = n;
   11 - 7 = n;
   16 - 8 = n;
   8 - 0 = n;
   10 - 3 = n;
   12 - 9 = n.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

2. If you know that 9 + 4 = 13 what subtraction facts do you know?

3. If you subtract 0 from any number, what is the result?

4. Here are some ideas in case you forget some facts. Complete these statements:
   (a) Because 16 - 8 = 8, 16 - 7 = _____.
   (b) Because 13 - 6 = 7, 13 - 5 = _____.
   (c) Because 15 - 8 = 7, 15 - 9 = _____.
   (d) Because 11 - 7 = 4, 11 - 8 = _____.

5. (a) You know that 14 - 7 = 7. How do you find 14 - 8?

(b) You know that 12 - 6 = 6. How do you find 12 - 5?

(c) You know that 15 - 9 = 6. How do you find 15 - 8?

6. Complete these statements:

(a) Because 18 - 10 = 8, 18 - 9 = _____.

(b) Because 16 - 10 = 6, 16 - 9 = _____.

(c) Because 13 - 10 = 3, 13 - 9 = _____.

(d) State a way to subtract 9 from any number.

7. (a) Because 13 - 8 = 5, 13 - 5 = _____.

(b) Because 11 - 4 = 7, 11 - 7 = _____.

(c) Because 15 - 6 = 9, 15 - 9 = _____.

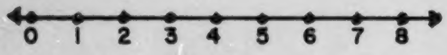
8. If you subtract 1 from any counting number, what is the result?

9. Using only the numbers 12, 5, and 7, state two additions and two subtractions.

10. BRAINTWISTER: The result of the operation of subtraction on a pair of numbers is 7. Write five of these pairs.

MATHEMATICAL SENTENCES USING THE NUMBER LINE

Using the Number Line



You remember that this is called a number line.

We can add 2 and 3 to get 5. "5 is greater than 2" and "5 is greater than 3". We write $5 > 2$ and $5 > 3$.

Since $5 > 2$, 5 is to the right of 2 on the number line and since $5 > 3$, 5 is to the right of 3 on the number line.

Suppose we add two whole numbers and neither number is 0. We get a sum. The sum is greater than either of the 2 numbers that we added. The sum is a number to the right of either of the two numbers on the number line.

"2 is less than 5" because we can add the number 3 to 2 to get 5 as the sum. 2 is to the left of 5 on the number line. We write $2 < 5$ and read it "2 is less than 5."

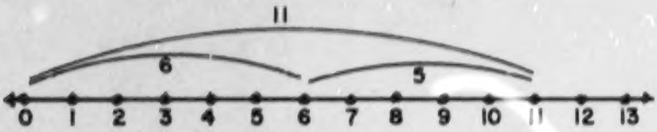
"3 is less than 5" because we can add the number 2 to 3 to get 5 as the sum. 3 is to the left of 5 on the number line. We write $3 < 5$ and read it "3 is less than 5".

Exercise Set 10

Copy each statement below. Write > or < in each blank so each mathematical sentence is true.

1. 8 (___) 6
2. 3 + 4 (___) 6
3. (11 + 9) (___) (11 + 8)
4. (30 + 20) (___) (31 + 21)
5. (53 + 40) (___) (53 + 41)
6. (200 + 800) (___) (200 +700)
7. (1,200 + 1,000) (___) (1,200 + 2,000)

NUMBER LINE PICTURES AND MATHEMATICAL SENTENCES



A number line can be used to suggest mathematical sentences. The two short curves above the number line suggest the two addends. The lone curve suggests the sum. The sentences suggested are

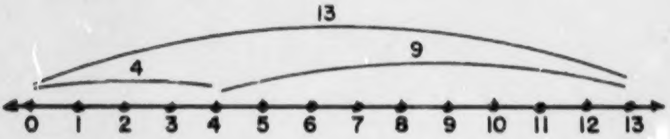
$$6 + 5 = 11$$

$$5 + 6 = 11$$

$$11 - 5 = 6$$

$$11 - 6 = 5$$

Exercise Set 11



1. Draw a number line as shown above. Draw curved lines to suggest this mathematical sentence $4 + 9 = n$.
   - (a) What is the sum represented by n?
   - (b) Is it larger than 4?
   - (c) Is it larger than 9?
   - (d) When two whole numbers are added, is the sum always larger than either addend?
   - (e) If your answer to (d) is, "No", give an example.

2. Draw a number line like the one in exercise 1. Use curves to suggest this subtraction: $8 - 3 = t$.

   (a) What is another name for the number represented by t?

   (b) Is the unknown addend, represented by t, larger than the sum?

   (c) Can an unknown addend ever be equal to the sum?

3. (a) How many units must be shown on a number line for you to picture this mathematical sentence, $12 - x = 8$?

   (b) Draw a number line. Use curves to suggest 12, the number represented by x, and 8 if $12 - x = 8$.

4. Draw number lines to suggest each of these mathematical sentences.

   (a) $7 + 4 = n$

   (b) $p + 5 = 9$

   (c) $8 + p = 11$

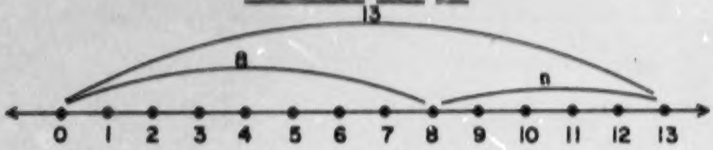
BRAINTWISTERS

5. How many units must be shown on a number line to picture this mathematical sentence, $140 - s = 40$?

6. How many units must be shown on a number line to picture this mathematical sentence, $p + 17 = 30$?

7. How many units must be shown on a number line to picture this mathematical sentence, $n - 29 = 13$?

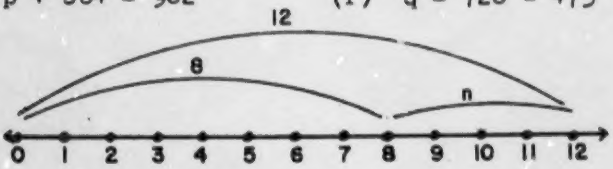
Exercise Set 12



1. This number line suggests $n + 8 = 13$ or $8 + n = 13$. Does it also suggest $n = 13 - 8$? To answer, "What number is the unknown addend n in $n + 8 = 13$?" can you think, "What number added to 8 is 13?" Can you also think, "n is 13 minus 8?"

2. Tell the operation to use to answer, "What number is the unknown addend so each of these mathematical sentences is true?" In each, tell which numbers are addends and which is the sum. Write your answers like this: (a) n and 7 addends; 18, sum

(a) $n + 7 = 18$	(d) $n = 436 - 194$
(b) $g = 3,649 - 1,856$	(e) $s + 824 = 1,726$
(c) $p + 364 = 982$	(f) $q = 728 - 475$

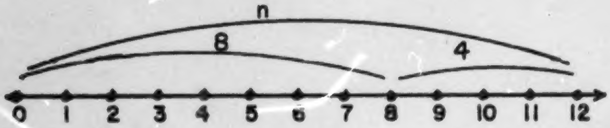


3. The above number line suggests $12 - n = 8$ or $12 - 8 = n$. Does it also suggest $8 + n = 12$? To answer, "What number is the unknown addend n in $12 - n = 8$?", can you think, "What number added to 8 is 12?" Can you also think, "n is 12 minus 8?"

4. Tell the operation to use to answer, "What number is the unknown addend so each of these mathematical sentences is true?"

(a) $15 - n = 11$ (c) $20 - p = 9$ (e) $13 - q = 8$

(b) $11 + n = 15$ (d) $9 + p = 20$ (f) $8 + q = 13$


n
8
4
0
1
2
3
4
5
6
7
8
9
10
11
12

5. The above number line pictures $n = 8 + 4$. Does it also suggest $n - 4 = 8$? Also $n - 8 = 4$? Also $4 + 8 = n$?

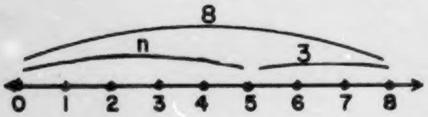
6. Tell the operation to use to answer, "What number is the sum so each of these mathematical sentences is true?".

(a) $x - 6 = 10$ (c) $y - 15 = 25$ (e) $z - 7 = 8$

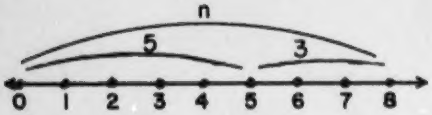
(b) $x = 10 + 6$ (d) $y = 15 + 25$ (f) $z = 7 + 8$

7. Write four mathematical sentences suggested by each picture.

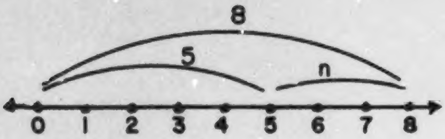
(a)


8
n
3
0
1
2
3
4
5
6
7
8

(b)


n
5
3
0
1
2
3
4
5
6
7
8

(c)



8. In each mathematical sentence tell which numbers are addends and which number is the sum. Then tell what operation you would use to find the number that p represents. (a) is done for you.

(a) $p = 13 - 7$

p and 7 are addends, 13 is the sum. Subtraction.

(b) $P + 800 = 1,743$

(c) $67 + p = 136$

(d) $p - 76 = 113$

(e) $p - 39 = 206$

(f) $p - 411 = 247$

(g) $813 - p = 542$

(h) $247 - p = 76$

(i) $320 - p = 106$

(j) $p - 40 = 630$

9. What number does p represent so each mathematical sentence is true?

(a) $10 + p = 30$

(b) $0 - p = 0$

(c) $0 = p + 0$

(d) $10 - p = 10$

(e) $p = 15 - 5$

(f) $15 - p = 15$

10. In which parts of exercise 9 is p not a counting number?

BRAINTWISTERS:

11. (a) Write a mathematical sentence using n, 12, and 15.

    (b) Write another mathematical sentence using n, 12, and 15. n is a different number than in exercise (a).

12. Is there a whole number to replace n so each of these mathematical sentences is true?

    (a) $20 - n = 30$ (b) $n + 30 = 20$

USING MATHEMATICAL SENTENCES IN PROBLEM SOLVING

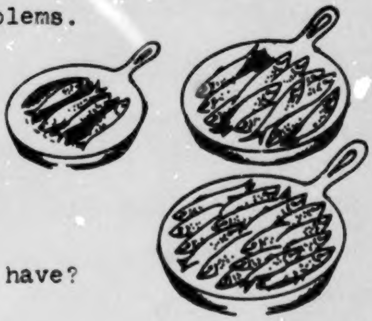
We have learned about the union of sets of objects.

We have learned about sets of objects within a set of objects.

How did the union of two sets help us understand addition?

How did sets within a set help us understand subtraction?

Now, let us see how we can use these ideas to help us answer questions in story problems.



1. Here is our first problem.
   Dick caught 4 fish.
   Jack caught 12 fish.
   Dick gave his fish to Jack.
   Then, how many fish does Jack have?

(a) Is the set of fish that Jack now has, the union of two sets of fish?

(b) Does this mathematical sentence fit the problem: $12 + 4 = n$? Why?

Does this mathematical sentence fit the problem: $4 + 12 = n$? Why?

(c) Now just think about the mathematical sentence: $12 + 4 = n$ or $4 + 12 = n$ How do you find n?

(d) We find that $n = 16$.

We can answer the question in our problem.

Jack now has 16 fish.

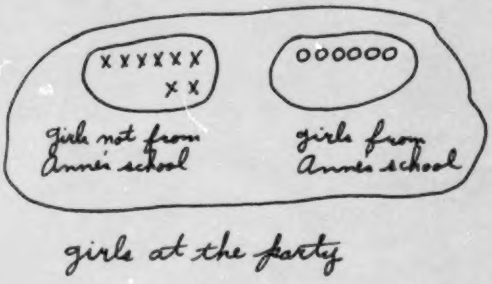
2. Here is our second problem.

Anne had a birthday party: 14 girls came to the party. 6 of the girls were from Anne's school. How many were not from her school?

(a) Is the set of girls at Anne's party the union of two sets?

(b) Do we know the number of girls in one of the sets?

(c) Can we represent the number of the girls in the other set by n?


xxxxxx
xx
oooooo
girls not from Anne's school
girls from Anne's school
girls at the party

(d) Does this mathematical sentence fit the problem: $14 - 6 = n$? or $n + 6 = 14$?

(e) What operation do we use to find n?
We can now find $n = 8$.

Summary

Mathematical sentences are helpful in problem solving. They help us to show number relationships in a short form. Here is a way you may use a mathematical sentence in solving a problem.

There are 22 children in a class. 10 of the children are girls. How many are boys?

$$10 + n = 22$$

$$\begin{array}{r} 22 \\ -10 \\ \hline 12 \end{array}$$

There are 12 boys in the class.

Writing a mathematical sentence we could also write:

$n = 22 - 10$, or

$22 - 10 = n$, or

$n + 10 = 22$

Finding n

Answering the question by writing an answer sentence.

Exercise Set 13

Find the answer for each of the following problems. Use the order of working suggested on page 101.

1. Ann practiced the piano 35 minutes on Friday. She practiced 40 minutes on Saturday. How many minutes did she practice on the two days?

2. Mary read two books. One book had 42 pages. The other had 26 pages. How many pages did she read in these books.

3. Nancy has 9 crayons in her box. The box will hold 12. How many more crayons does she need to fill the box?

4. In a fish pond there are 25 black fish and 20 gold fish. How many fish are in the pond?

5. Jim has a paper route. He has delivered 35 of his 49 papers. How many more papers does he have to deliver?

6. There were 25 girls at a party. 15 of them were watching television. The others were playing. How many girls were playing?

7. Jack has 59 stamps in two envelopes. In one envelope there are 24 stamps. How many stamps are there in the other envelope?

8. Sue is saving to buy a book that costs 98 cents. She has 75 cents. How much more money does she have to save to buy the book?

9. Tom spelled correctly 16 words on a test. He spelled 20 words correctly on another test. How many words did Tom spell correctly on both tests?

10. Our auditorium was decorated with red balloons and white balloons. There were 63 balloons in all. If 41 balloons were red, how many were white?

11. At a popcorn sale, 29 bags were sold in one day. If 12 bags were sold in the morning, how many bags were sold in the afternoon?

12. David weighs 60 pounds. His little brother weighs 20 pounds. How many pounds do they weigh together?

BRAINTWISTERS

13. There are 20 pupils in a class. The class has just two committees to plan a party. The refreshments committee has 7 members. The games committee has 5 members. James, Mary, and Bob are on both committees.
How many pupils are on just one committee?
How many pupils are on either one or two committees?
How many pupils in the class are not on any committee?

14. 300 pupils attended the school football game on Tuesday. 250 pupils attended the school football game on Wednesday. 50 pupils attended both the games.

How many pupils attended just one of the games?

How many pupils attended the game on Wednesday that did not attend the game on Tuesday?

DOING AND UNDOING - ADDITION AND SUBTRACTION

There are many actions that undo other actions. For example, Jack found a dime. He lost that dime.

1. Complete the chart below with the missing "doing" or "undoing" actions.

Doing	Undoing
1. stand	sit
2. open	close
3. dress	____
4. ____	untie
5. button	____

2. Give some sentences like this one which tell about both doing and undoing:

"Ellen opened her book and then closed it."

We have seen that one action may undo another action. This exercise will help us to see if subtracting a number will undo adding that same number.

3. (a) Think of 8. Add 3 and then subtract 3.
What is your result?
Finish this sentence: (8 + 3) - = ____ .

(b) Think of 10. Subtract 6 and then add 6.
What is your result?
Finish this sentence: (10 - 6) + 6 = ____.

(c) What must you do to $6 + 4$ to get 6?

(d) What must you do to $6 - 4$ to get 6?

Adding a number and subtracting that same number undo each other. For example, if we start with 9, then add 2, and then subtract 2, the result is 9, the number we started with. Subtracting 2 undid adding 2. We can write

$$(9 + 2) - 2 = 9.$$

Subtracting a number and adding that same number undo each other. For example, if we start with 5, then subtract 3, and then add 3, the result is 5, the number we started with. Adding 3 undid subtracting 3. We can write

$$(5 - 3) + 3 = 5.$$

Exercise Set 14

Answer Yes or No for exercises 1 - 3

1. Is the result the same number for (a) and (b) below?
   (a) Start with 8, add 2, and then subtract 2.
   (b) Start with 8 and add 0.
   (c) Is $(8 + 2) - 2 = 8 + 0$?

2. Is the result the same number for (a) and (b) below?
   (a) Start with 14, subtract 6, and then add 6.
   (b) Start with 14 and add 0.
   (c) Is $(14 - 6) + 6 = 14 + 0$?

3. Is the result the same number for (a) and (b) below?
   (a) Start with n, add 6, and then subtract 6.
   (b) Start with n, and add 0.
   (c) Is $(n + 6) - 6 = n + 0$?

4. Write mathematical sentences for (a) and (b).
   (a) Start with 9, add 5, and then subtract 5.
   (b) Start with 9, and add zero.
   (c) Are the results the same for (a) and (b)?

5. Write mathematical sentences for (a) and (b).

   (a) Start with n, add 7, and then subtract 7.

   (b) Start with n, and add zero.

   (c) Are the results the same for (a) and (b)?

6. Is it true that if you start with any whole number and add 0, the result is that whole number?

   Give some examples.

7. Apply the "undoing" idea to these operations.

   Exercises (a) and (b) are done for you.

	DO	UNDO
(a)	$5 + 2$	$5 + 2 - 2$
(b)	$6 - 4$	$6 - 4 + 4$
(c)	$5 + 3$	
(d)	$18 - 10$	
(e)	$25 + 20$	
(f)	$3 + n$	
(g)	$n - 2$	
(h)	$p - n$	

8. What operation is used to find n in each of these true mathematical sentences?

   (a) $5 + 6 = n$

   (b) $n = 7 - 4$

   (c) $5 + n = 6$

   (d) $n + 2 = 43$

   (e) $n = 32 + 65$

   (f) $91 - 60 = n$

   (g) $75 = n + 31$

   (h) $83 + 81 = n$

   (i) $126 - 100 = n$

9. What number is n for each of the exercises (a) to (i) in exercise 8?

10. **BRAINTWISTER:** The mathematician calls 0 the identity element for addition. What is the meaning of identity? What do you think identity element for additon means?

11. **BRAINTWISTER:** (a) Two numbers to be operated on are 8 and (10 - 10). The operation is addition. What is the result?

    (b) In exercise (a) if (10 - 10) is replaced by 0, what is the result?

    (c) Two whole numbers to be operated on are a and (6 - 6). The operation is addition. What is the result?

    (d) Two whole numbers to be operated on are m and (n - n). The operation is addition. What is the result?

12. **BRAINTWISTER:** In exercise 11, replace the word addition by the word subtraction. Answer each of the four parts of exercise 11.

Exercise Set 15

Write a mathematical sentence for each of these problems. Then solve.

1. (a) Tom and Peter had 72 cookies for their class picnic. The boys ate 12 on their way to the picnic. How many were left for the picnic?

   (b) Peter's mother had to bring one dozen more cookies for the class picnic. How many cookies were there for the class?

   (c) Show how the operation of exercise (b) undoes the operation of exercise (a) by use of a mathematical sentence.

2. (a) Margaret has 12 addition exercises to do. Her teacher gave her 3 more exercises.

   (b) Margaret does 3 exercises.

   (c) Show how the operation part (b) undoes the operation of part (a) by use of a mathematical sentence.

3. (a) Jon had $45 in the bank. He spent $20 during the summer for swimming lessons.

   (b) Jon earned $ 20 and put it in the bank.

   (c) Show how the operation of part (b) undoes the operation of part (a) by use of a mathematical sentence.

MORE ABOUT ADDITION AND SUBTRACTION OF WHOLE NUMBERS

1. Think of the set of the first 5 whole numbers.

   A = {0, 1, 2, 3, 4}

   (a) Add any 2 of these numbers. You are permitted to add one of these numbers to itself. What numbers do you get for sums?

   (b) Is each of the sums in Set A?

   (c) Which sums are in Set A?

   (d) Why are the other sums not in Set A?

2. Suppose you have just the numbers in Set S.

   S = {8, 9, 10, 11}.

   Find the sum of any two numbers in Set S. Are any of these sums members of Set S?

   Is Set S closed under addition?

3. If two whole numbers are added, is the result always a whole number? Try some examples.

4. If two whole numbers are subtracted, is the result always a whole number? Try some examples. Do you sometimes get a whole number?

It is always possible to add two whole numbers because there is always a whole number to use as a sum. This means the set of whole numbers is closed under addition.

It is not always possible to subtract two whole numbers because there is not always a whole number to use as the other addend. There is no whole number n so that $3 - 5 = n$. This means that the set of whole numbers is not closed under subtraction.

Exercise Set 16

1. Pretend you know only the set of even whole numbers. Write a few of the members of the set.

   (a) Choose six pairs from the set. Add them using a form such as shown at the right. (Remember that you can use only the even whole numbers as addends.)

   $$\begin{array}{rl} 26 & \text{even} \\ \underline{14} & \text{even} \\ 40 & \text{even} \end{array}$$

   (b) What can you say about the sum for each addition you tried?

   (c) When you add two even whole numbers, do you expect to get a sum which is always odd? Always even? Sometimes odd and sometimes even?

   (d) How many pairs of numbers did you try? Try enough pairs so that you are sure of your answer in (c).

   (e) Is the sum of any two even whole numbers a number in the set of even whole numbers? Is the set of even whole numbers closed under addition?

2. **BRAINTWISTER:** Set A = {0, 3, 6, 9, 12, 15, 18, 21, ...} Think of all pairs of Set A such as 0 and 3, 3 and 3, 6 and 9, and so on. Think of the sum of each pair. Call this set of sums Set B. Write Set B. Is every member of Set B a member of Set A?

3. Think of the set of odd whole numbers. Write a few members of the set.

   (a) Choose six pairs from the set. Add them. Use a form such as shown at the right. (Remember that you can use only the odd whole numbers as addends.)

   11 odd
   19 odd
   30 even

   (b) You have only odd numbers in this set. Is there a number to use for a sum in each pair you choose? Is the sum an odd whole number?

   (c) When you add a pair of odd numbers, do you expect to get a result which is always odd? Always even? Sometimes odd and sometimes even?

   (d) How many pairs of numbers did you try? Try enough pairs so that you are sure of your answer in (c).

   (e) Is the sum of any two odd whole numbers a number within the set of odd whole numbers? Is the set of all odd whole numbers closed under addition?

4. **BRAINTWISTER:** Set A = {1, 4, 7, 10, 13, 16, 19, 22, ...} Think of some pairs of Set A such as 1 and 4, 4 and 10, 7 and 7, and so on. Think of the sum of each pair. Call this set of sums Set B. Write Set B. Is any member of Set B a member of Set A? Is Set A closed under addition?

Exercise Set 17

BRAINTWISTER SET

1. Pretend you know only the set of numbers {1, 2, 3, 4}. This means that in this exercise you may use only the numbers 1, 2, 3, 4.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

   (b) Did you fill in each space of the chart?

   (c) If not, why not?

+	1	2	3	4
1				
2				
3				
4				

2. Pretend you know only the set of numbers {6, 8, 10, 12, 14}.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

   (b) Did you fill in each space of the chart?

   (c) If not, why not?

+	6	8	10	12	14
6					
8					
10					
12					
14					

3. Pretend you know only the set of numbers {1, 3, 5, 7}.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

   (b) Did you fill in any space of the chart?

   (c) If not, why not?

+	1	3	5	7
1				
3				
5				
7				

4. (a) In exercise 1 is the set closed under addition?

   (b) In exercise 2 is the set closed under addition?

   (c) In exercise 3 is the set closed under addition?

MORE PROBLEM SOLVING

Solving written problems is an exercise in careful thinking. You may need to read a problem several times. Be sure you understand what question is asked in the problem.

Look for the statements in the problem that give you information. This information may be in more than one statement.

Write a mathematical sentence to show the relationships in the problem. (You may wish to look at pages 99 - 101.)

Study the mathematical sentence and decide what operation to use. Then carry out the operation.

Write an answer sentence to explain that your answer from the operation is an answer to the question in the problem.

Example.

Jack has 438 stamps in his collection. Bill has 326 stamps in his collection. Jack has how many more stamps than Bill?

What mathematical sentence shows the relationships in the problem?

438 - 326 or 326 + n = 438

What operation do we use to find n?

Is n = 112?

The answer sentence is "Jack has 112 more stamps than Bill."

Exercise Set 18

1. During the summer the Smith family traveled 2,140 miles on their way to Yellowstone Park. They traveled 2,037 miles on their way home. How many miles did they travel altogether?
2. Margaret has read 113 pages. Her book has 247 pages. How many more pages has she left to read?
3. The Stone family car has traveled 12,547 miles. The Brown family car has traveled 11,325 miles. How much farther has the Stone family car traveled than the Brown family car?

4. How much larger is three thousand, two hundred seventy-five than two thousand, one hundred thirty-three?

5. Mary and her mother collect old buttons. Mary's mother has 275. Mary has just begun her collection and has 124. How many buttons do Mary and her mother have in all?

6. The Grant School has 225 Red Cross boxes. They filled 114 of theirs. How many more boxes are to be filled?

7. On Saturday, 1,462 people bought tickets for the Little League World Series games. On Sunday, 2,526 people bought tickets for the game. How many tickets were bought for the games on these two days?

8. Helen was born in 1950. How old will she be on her birthday this year?

THE ASSOCIATIVE PROPERTY FOR ADDITION

1. Suppose you were told to add 6, 7, and 5.

   (a) Can you add three numbers at the same time?

   (b) We can add 6 and 7 because we can add just two numbers. Let's write (6 + 7) + 5. This means we will add 6 and 7. We get 13. Then we add 13 and 5 and get 18. We have the sum of the three numbers 6, 7, 5. How many numbers did we add at any one time?

   (c) We can also write 6 + (7 + 5). This means we will add 7 and 5. We get 12. Then we add 6 and 12.

   (d) Are the final sums in (b) and (c) the same?

It is not possible to add more than two numbers at a time. If we have more than two numbers to add, we must group just two numbers. For example, if we want to add 536, 451, and 612, we cannot do all three of them in one operation. We can add 536 and 451 and then add 612 to this sum. Or we can add 451 and 612, and then add this sum to 536. We could write (536 + 451) + 612 or 536 + (451 + 612) to show how we add the three numbers.

In the example at the top of this page we must add just two numbers at a time. We must group just two numbers together. To do this for 6 + 7 + 5 we can write

$$(6 + 7) + 5 = n.$$

The parentheses mean that we are grouping the 6 and 7 and we think of 6 + 7 as one number, 13. Then the sum is 13 + 5 or 18. We could write

$$6 + (7 + 5).$$

This means we are grouping the 7 and 5 and we think of this as one number, 12. Then the sum is 6 + 12 or 18. (6 + 7) + 5 and 6 + (7 + 5) are each names for the same number, 18. The way in which we grouped the numbers did not change the sum. When we group 6 + 7 + 5 as (6 + 7) + 5 or as 6 + (7 + 5), we are using the associative property for addition. We must group the numbers by twos since we can add just two numbers at a time.

2. If we use the associative property for addition to write $3 + 2 + 4 = n$, we would write:

$$(3 + 2) + 4 = n$$
$$5 + 4 = n$$
$$9 = n$$

or

$$3 + (2 + 4) = n$$
$$3 + 6 = n$$
$$9 = n$$

Find each sum. Use the associative property for addition as was done above.

(a) $2 + 1 + 5 = n$
(b) $6 + 3 + 2 = n$
(c) $8 + 2 + 3 = n$

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3. (a) Tell how to do this operation: $(6 + 7) + 5$.

(b) Tell how to do this operation: $6 + (7 + 5)$.

(c) Why must we group two of the numbers in adding 6, 7, 5?

4. (a) What is the result of $(6 + 7) + 5$?

(b) What is the result of $6 + (7 + 5)$?

(c) Is $(6 + 7) + 5 = 6 + (7 + 5)$?

5. (a) Is $(3 + 4) + 5 = 3 + (4 + 5)$?

(b) Are $(3 + 4) + 5$ and $3 + (4 + 5)$ different names for the same number?

(c) In what way is $(3 + 4) + 5$ different from $3 + (4 + 5)$?

Summary

Adding of three numbers must be done in two steps. You may add 63, 24, and 82 in either of two ways if the order is not changed.

$$(63 + 24) + 82 = 87 + 82 = 169$$

$$63 + (24 + 82) = 63 + 106 = 169$$

The sum is the same even if we did group the addends differently.

So, we can write

$$(63 + 24) + 82 = 63 + (24 + 82).$$

Exercise Set 19

1. The associative property is used in finding the sum of 14 and 5. You have not called it by this name but you have used it:

$$\begin{aligned} 14 + 5 &= (10 + 4) + 5 && \text{Step I} \\ &= 10 + (4 + 5) && \text{Step II} \\ &= 19 && \text{Step III} \end{aligned}$$

(a) Is $(10 + 4) + 5 = 10 + (4 + 5)$ illustrative of the associative property for addition?

(b) Is 19 a different name for $10 + (4 + 5)$?

2. The associative property can help you do some additions easier. These are ways of adding $15 + 9 + 11$.

(a)

$$\begin{aligned} 15 + 9 + 11 &= (15 + 9) + 11 \\ &= 24 + 11 \\ &= 35 \end{aligned}$$

(b)

$$\begin{aligned} 15 + 9 + 11 &= 15 + (9 + 11) \\ &= 15 + 20 \\ &= 35 \end{aligned}$$

Is the sum the same using either method (a) or (b)? Many pupils like method (b) better when they add without paper and pencil. Why?

3. What property of addition is illustrated by $4 + (1 + 6) = (4 + 1) + 6$?

4. What properties of addition are illustrated by $(3 + 5) + 8 = 3 + (8 + 5)$?

5. (a) Does (5 - 3) - 2 = 5 - (3 - 2)?

(b) Does (9 - 4) - 3 = 9 - (4 - 3)?

(c) Is subtraction an associative operation?

6. To find the sum 4 + 2 + 7, a boy wrote 4 + 2 = 6 + 7=13. The statement he wrote is wrong. Why?

For exercises 7 - 11 write the correct words, numerals and mathematical sentences to complete this chart. Remember that numerals in parentheses name one number.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
7.	87, 56	___	Subtraction	___
8.	(8 + 6), 5	19	___	___
9.	11, (9 + 6)	___	Addition	___
10.	(6 + 5), (8 + 4)	___	Addition	___
11.	27, (8 + 8)	11	___	___

12. Write + or - in each blank so each sentence is true.

(a) (12 ___ 8) ___ 7 = 13

(b) 13 ___ (8 ___ 6) = 15

(c) (42 ___ 61) ___ 52 = 155

(d) 54 ___ (38 ___ 11) = 5

13. Write parentheses to help you do these.

(a) 4 + 6 + 19

(b) 11 + 9 + 25

(c) 17 + 3 + 29

(d) 59 + 12 + 8

REVIEW

Exercise Set 20

Some of the properties that you have studied are reviewed below. Tell if the statements are always true, sometimes true, or never true. Give examples.

Property Stated for Addition	Property Stated for Subtraction
1. (a) If 0 is added to a whole number, the result is that whole number.	1. (b) If 0 is subtracted from a whole number, the result is that whole number.
2. (a) If two whole numbers are added, the result is a whole number.	2. (b) If two whole numbers are subtracted, the result is a whole number.
3. (a) If the order of adding two whole numbers is changed, the sum is unchanged.	3. (b) If the order of subtracting two whole numbers is changed, the unknown addend is unchanged.

4. Find n so each mathematical sentence is true.

(a) $6 + n = 6 + 9$

(b) $12 + n = 14 + 12$

(c) $n + 21 = 21 + 42$

(d) $(8 + 8) + n = 8 + (9 + 4)$

(e) $(12 + 8) - n = 12 + (8 - 3)$

(f) $(3 + 5) + (n + 2) = (6 + 2) + (3 + 5)$

5. Place parentheses in $18 - 12 + 2$ so the result is 4.
Is $(18 - 12) + 2 = 4$? Is $18 - (12 + 2) = 4$?

6. Place parentheses so
   (a) $26 - 12 + 9 = 5$ (c) $26 - 12 + 9 = 23$
   (b) $57 - 37 - 20 = 0$ (d) $57 - 37 - 20 = 40$

7. On your paper, write the letter which is beside each true mathematical sentence.
   (a) $15 + 19 = 19 + 15$ (d) $18 + 6 = 6 + 18$
   (b) $15 + 19 > 15 + 18$ (e) $18 + 6 > 6 + 18$
   (c) $15 + 19 < 16 + 19$ (f) $18 + 6 < 18 + 7$
   (g) $118 + 394 = 493 + 811$
   (h) $118 + 394 = 394 + 118$

8. Is $(36 + 75) + 19 = 19 + (36 + 75)$? Without adding the mathematical sentence is true by using the commutative property for addition. Do you see that $(36 + 75)$ and 19 are the two addends? Tell which property is illustrated in each of these.
   (a) $2 + (3 + 4) = (2 + 3) + 4$
   (b) $(18 + 19) + (39 + 12) = (39 + 12) + (18 + 19)$
   (c) $(8 + 9) + 6 = 6 + (8 + 9)$
   (d) $(8 + 9) + 6 = (9 + 8) + 6$
   (e) $(8 + 9) + 6 = 8 + (9 + 6)$

9. (a) Does (2 + 7) - 2 = 2 + (7 - 2)?

(b) Does 2 + (7 - 2) = 7 + (2 - 2)?

(c) Does (2 + 7) - 2 = 7 + (2 - 2)?

10. Copy the letters (a) to (d) on your paper.

Beside each letter, write the correct words, numerals, and mathematical sentences needed to complete this chart.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
(a)	(7 + 4), 8	3	______	______
(b)	20, (40 + 30)	___	Addition	______
(c)	(30 - 30), (40 - 40)	___	Subtraction	______
(d)	(83 - 61), (199 + 1)	___	Addition	______

11. Writing parentheses helps you do the addition, 23 + 19 + 11 without paper and pencil. See the box on the right. Write parentheses to help you do these.

23 + 19 + 11 = n
23 + (19 + 11) = n
23 + 30 = n
53 = n

(a) 12 + 8 + 16

(b) 87 + 8 + 92

(c) 18 + 982 + 767

(d) 13 + 17 + 26

(e) 78 + 36 + 14

(f) 59 + 11 + 68

12. Find what number n represents so that each mathematical sentence is true. Be careful. There may be no answer, one answer, or even more than one answer.

(a) $(3 + 2) + 8 = n$

(b) $(3 + 2) + n = 8$

(c) $(3 + 2) - n = 8$

(d) $1 + n = n$

(e) $2 + n = n$

(f) $3 + n = n$

(g) $n = n$

(h) $1 + n = n + 1$

(i) $7 + n = n + 7$

13. Does $(6 - 4) - 1 = 6 - (4 + 1)$? Why?

14. Does $(6 - 4) - 1 = 6 - (4 - 1)$? Why?

15. BRAINTWISTER: Make up two examples like exercise 13 and exercise 14. Which one is true and which one is false?

16. BRAINTWISTER: How many counting numbers are greater than 19 and less than 25? Is $n = 25 - 19$ the correct relationship for this problem?

Exercise Set 21

Write a question that requires addition for each of exercises 1 to 3.

1. Tom and Bob were collecting old clothes for the church drive. Bob worked for 45 minutes on Monday and 30 minutes on Tuesday.

2. Tom called at 12 houses and Bob called at 17 houses.

3. Bob collected 15 coats and 21 dresses.

4. Write a question that requires subtraction for each of the exercises 1 to 3.

For each of the exercises 5 through 7 write: (a) the numbers operated on; (b) the mathematical sentence; (c) the result. Arrange your work in a chart like this.

	Numbers Operated On	Mathematical Sentence	Result
5.	________	____________	__
6.	________	____________	__
7.	________	____________	__

5. How much less than a dozen is 8?

6. Bob had a collection of 37 toy airplanes. One day he could find only 23. How many were gone?

7. Five of the 28 girls on the playground went back early into the school building. How many remained on the playground?

Exercise Set 22

1. Study this: $(6 - n) - 1 = 6 - (n + 1)$

   (a) What is the largest whole number n can be?

   (b) What is the smallest whole number n can be?

   (c) Find all the whole numbers n can be.

2. BRAINTWISTER: Tell what whole number n is so that each mathematical sentence is true. Be careful. There may be no answer, one answer, or more than one answer.

   (a) $n - 1 = 1 - n$

   (b) $n - 10 = 10 - n$

   (c) $6 + n = n + 6$

   (d) $n + 50 = 50 + n$

   (e) $n = n - 1$

   (f) $10 - n = n$

Exercise Set 23

BRAINTWISTER SET

Pretend you have two sets, A and B. Set A has 8 members. Set B has 5 members. The intersection of Set A and Set B is Set C. Set C has 2 members.

1. Make a drawing of these intersecting sets.

2. How many members in Set A are not in Set B?

3. How many members are in Set B and not in Set A?

4. How many members are in the union of Set A and Set B?

5. Write the mathematical sentence from which you got the answer to exercise 4.

Pretend you have two sets, D and E. Set D has 8 members. Set E has 5 members. The intersection of Set D and Set E is Set F. Set F is an empty set.

6. Make a drawing of these intersecting sets.

7. How many members in Set D are not in Set E?

8. How many members in Set E are not in D?

9. How many members are in the union of Set D and E?

10. Write the mathematical sentence from which you got the answer to exercise 9.

11. **SUPER BRAINTWISTER:** At Grant School there is a Mathematics Club. The members of the club are also members of certain sets. The members of Set A have read the magazine, Popular Mathematics. The members of Set B have read the book, Mathematics is Fun. The members of Set C have read both the book and the magazine. Set A has 6 members, Set B has 5 members, and Set C has 3 members.

    (a) Are all members of Set C also members of Set A?

    (b) Are all members of Set C also members of Set B?

    (c) How many members of Set C are also members of Set A?

    (d) How many members of Set C are also members of Set B?

    (e) The intersection of Set A and Set B has how many members?

    (f) Make a drawing of the intersecting sets.

    (g) How many members are there in the club?

Chapter 4

PROPERTIES OF MULTIPLICATION AND DIVISION

ARRAYS

Exploration

John went to the ice cream shop to buy cake and ice cream. The shop sells 3 kinds of cake, chocolate, angel food, and coconut; and 4 flavors of ice cream, vanilla, chocolate, strawberry, and cherry. How many choices of one kind of cake and one kind of ice cream does he have? Copy the following headings on the chalkboard and make a chart of his choices.

	Vanilla Ice Cream	Chocolate Ice Cream	Strawberry Ice Cream	Cherry Ice Cream
Chocolate Cake				
Angel Food Cake				
Coconut Cake				

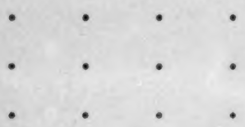
Use the chart to answer these questions.

How many choices does he have with 1 kind of cake and 4 flavors of ice cream?

How many choices does he have with 2 kinds of cake and 4 flavors of ice cream?

How many choices does he have with 3 kinds of cake and 4 kinds of ice cream?

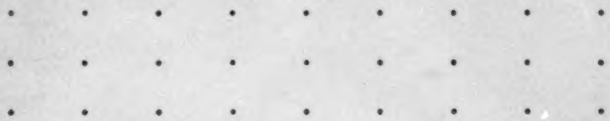
Could we make a chart of his choices without using the words cake and ice cream and letting dots stand for the choices? Would the chart look like this?



If peppermint ice cream were added to his choices of flavors of ice cream, how many choices of three kinds of cake and five flavors of ice cream does he have? Make a chart to answer the question.

Make other charts to show that John has 6 choices of ice cream flavors, 7 choices of ice cream flavors, 8 choices of ice cream flavors, 9 choices of ice cream flavors. From these charts what questions can you answer? How are the charts alike? How are the charts different? Can you use one chart to answer all the questions about John's choices?

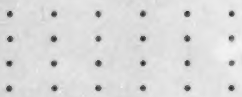
Did your chart look like this?



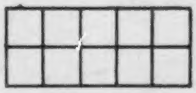
This chart is called an array. It is an orderly arrangement of objects in rows and columns. In this array there are 3 rows and 9 columns. There are 27 elements in the array.

What arrays are shown below?

(a)



(b)



How many rows are in array (a)?

How many columns are in array (a)?

How many rows are in array (b)?

How many columns are in array (b)?

How many elements are in each array?

We have seen that an array can be used to show all matchings of one set with another. Arrays are useful in other situations. Some sets of objects are themselves arranged in arrays. Here are some examples:

a) eggs in a carton

b) panes of glass in a window or door

c) seats in an auditorium

d) pieces of candy in a box

e) crayons in a box

f) linoleum blocks on our room floor, etc.

Can you think of some sets of objects arranged in arrays to add to the list of examples?

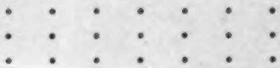
Draw an array of dots which represents a possible arrangement of 15 panes in a window. How many panes are there in a window whose panes form a 3 by 4 array?

Exercise Set 1

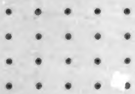
1. Copy the following table. Use the arrays shown below to help you complete the table.

Exercise	Rows	Columns
Example:	3	7
a.		
b.		
c.		
d.		
e.		
f.		

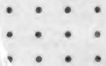
Example:



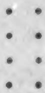
a.



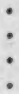
b.



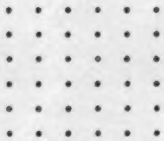
c.



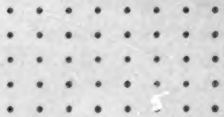
d.



e.



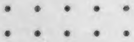
f.



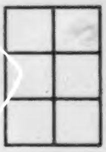
2. Copy and fill in the table below.

Exercise	Rows	Columns	Elements
Example:	2	5	10
a.			
b.			
c.			
d.			
e.			

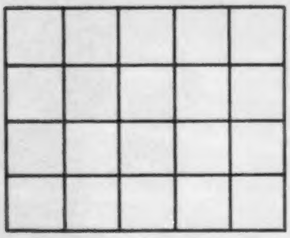
Example:



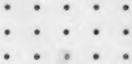
a.



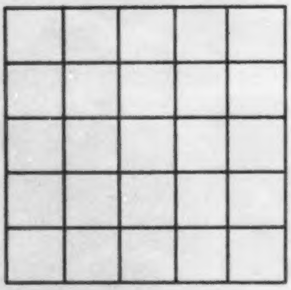
b.



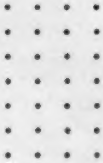
c.



d.



e.



Exercise Set 2

1. Draw an array that has 2 rows and 4 columns.

For each of the following pairs of numbers, draw an array. The first number tells the number of rows. The second number tells the number of columns. Under each array, write the number of elements in the array.

2. 5 , 2

3. 3 , 4

4. 3 , 3

5. 5 , 3

6. 4 , 2

7. 3 , 5

8. Draw an array of 15 elements that has 3 rows.

9. Draw an array that has 10 elements and 5 columns.

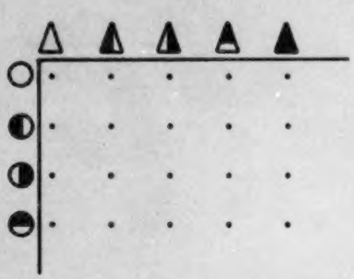
10. Draw an array that has 16 elements and 4 rows.

MULTIPLICATION

An operation on numbers is a way of thinking about two numbers and getting one and only one number. When we think of 4 and 5 and get 20, we call this multiplication.

We use arrays to help us understand multiplication.

Where is an array which shows all the matchings of a set of 4 elements with a set of 5 elements? There are 20 matchings.



Some examples of these matchings are:

○ △, ◐ ▲, ⊖ ▲, ◑ ▲.

Notice 2 objects are needed to make 1 matching. Try to draw some other possible matchings.

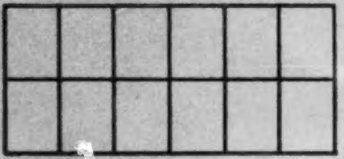
The array gives us a picture of the mathematical sentence, 4 × 5 = 20. We read this "4 times 5 equals 20." We write the number of rows first and the number of columns second.

$$\underset{\substack{\uparrow \\ \text{rows}}}{4} \times \underset{\substack{\uparrow \\ \text{columns}}}{5} = \underset{\substack{\uparrow \\ \text{elements}}}{20}$$

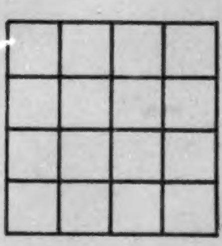
Exercise Set 3

Write the mathematical sentence which belongs with each of these arrays.

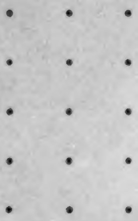
1.



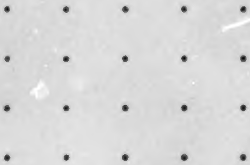
2.



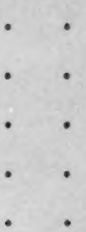
3.



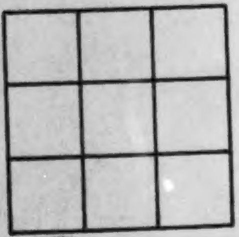
4.



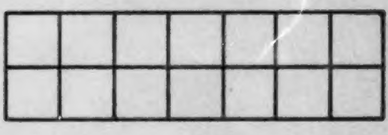
5.



6.



7.

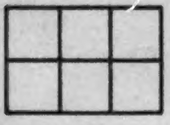


Exercise Set 4

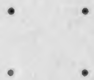
1. Each of 4 elementary schools has a basketball team of 5 players. Draw an array with one dot for each player. Using an array describe the number of players. Write the mathematical sentence. What does this sentence say about the total number of players on the team?

Make a story problem which each of the following arrays helps to answer.

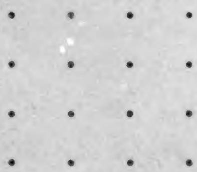
2.



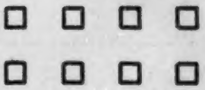
3.

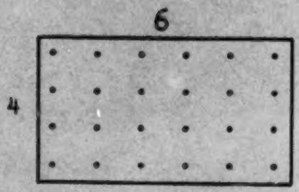


4.

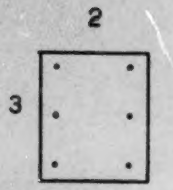


5.

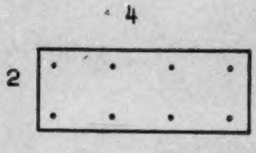




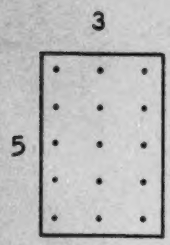
A



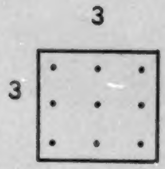
B



C



D



E

6. Three teams of Cole School are to play. Three teams from Grant School are to play. Which array above shows the number of games that can be arranged if each Cole team plays each Grant team just once?

7. Mr. Smith is buying a car. The company offers tops in 5 colors and bodies in 3 colors. Which array above shows the number of possible color combinations?

8. Mary has 4 dolls and 6 dresses. Which array above shows how many matchings of dolls and dresses Mary can make?

9. Three children play violin and two other children play piano. Which array above shows how many duets can be played so that each violinist plays with each pianist?

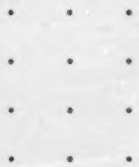
10. Sue has 2 watches and 5 different colored bands. Make an array to show how many ways Sue can match her watches and bands.

    a. How many rows must the array have?
    b. How many columns must the array have?
    c. How many ways can Sue match her watches and bands?

11. Paul has 2 neckties and 3 colored handkerchiefs. Draw an array to show the different ways he can match the neckties and handkerchiefs.

    a. How many rows has the array?
    b. How many columns has the array?
    c. How many matchings could Paul make?

12. Linda has some necklaces and some bracelets. Here is an array which shows all the ways she can match her necklaces and her bracelets.



    a. If Linda has 4 necklaces, how many bracelets has she?
    b. Write a mathematical sentence which belongs with the array.
    c. How many ways can Linda match her necklaces and bracelets?

Exercise Set 5

Write a mathematical sentence which goes with each problem. Draw an array if you need one. Beginning with problem 4, be sure you answer each question in a complete sentence.

1. Write the mathematical sentence which shows the matchings of a set of 2 things with a set of 9 things.

2. Write the mathematical sentence which shows the matchings of a set of 2 objects and a set of 3 objects.

3. In how many arrays can 12 dots be arranged? Write the mathematical sentences.

4. The calendar is arranged in 5 rows of squares. Each row is divided into 7 squares. How many squares are shown on the calendar?

5. Some Christmas ornaments were packed in boxes of 4 rows. There were 3 ornaments in each row. How many ornaments were there in the box?

6. A bar of chocolate candy was divided into 2 rows of 4 squares each. How many squares of chocolate were in the bar?

7. There are 3 rows of windows in our room. Each row has 5 windows. How many windows are there in our room?

8. Candy was arranged in a box in 5 rows with 9 pieces of candy in each row. How many pieces of candy were in the box?

9. For our class picture, the children were grouped in 4 rows. There were 8 children in each row. How many children were there in the picture?

10. In a box, there were 2 rows of erasers, with 2 erasers in each row. How many erasers were there in the box?

BRAINTWISTER: How many possible arrays of 24 dots could you make? (Draw them if necessary.) Describe each array by writing a mathematical sentence.

HOW TO SHOW MULTIPLICATION

Exploration

When we think, talk, and write about the operation of addition, we have certain ways of indicating addition with words and other symbols.

Write a mathematical sentence which shows the addition of 4 and 5.

How do you read this mathematical sentence?

What are the numbers 4 and 5 called?

What is the number 9 called?

The sum, 9 , is the result of operating on the addends, 4 and 5.

There is also a mathematical sentence to indicate multiplication. If the two numbers operated on are 4 and 5 and the result is 20, we can write the mathematical sentence

$$4 \times 5 = 20.$$

The numbers 4 and 5 are called *factors* of 20. The number 20 is called the *product* of 4 and 5.

The product, 20 , is the result of operating on the factors 4 and 5.

Compare these two mathematical sentences.

a) $4 + 5 = s$ b) $4 \times 5 = p$

How are the sentences alike?

How are the sentences different?

It is confusing to call the parts of the sentences by the same names because the operations are different. The 4 and 5 of sentence (b) must have special names. What are they called? What is 20 called?

Here are some other mathematical sentences.

c) $5 \times 8 = 40$ d) $36 \times 424 = p$

What do we call the 5 in sentence (c)?

What is the 8 called?

What is the 40 called?

When we operate on two factors and get a product, we multiply.

What are the two factors in sentence (d)?

What is the product?

What number is the name of the product p ?

We do not know now, but soon we will learn how to find it.

Summary

We write a multiplication sentence like this:

$5 \times 4 = 20.$

We read a multiplication sentence like this:

5 times 4 is equal to 20.

5 times 4 equals 20.

The names of the parts of a multiplication sentence are:

5	×	4	=	20
↑		↑		↑
factor	times	factor	equals	product .

When we operate on two factors and get a product, we multiply.

USING ARRAYS IN MULTIPLICATION

Exercise Set 6

Each pair of numbers listed below shows the number of rows and the number of columns in an array. Write the mathematical sentence which tells how many elements are in each array.

Example: 5 , 3 5 × 3 = 15

1. 4 , 2
2. 4 , 3
3. 4 , 4
4. 5 , 4
5. 6 , 4
6. 7 , 4
7. 8 , 4
8. 9 , 4
9. 2 , 5
10. 3 , 5
11. 4 , 5
12. 5 , 5
13. 5 , 6
14. 5 , 7
15. 5 , 8
16. 5 , 9

Make a chart with two columns as shown below. Complete your chart. An example is given.

Number of Rows and Number of Columns in Each Array		Mathematical Sentence Which Describes the Number of Elements in Each Array
Example	5 , 2	5 × 2 = 10
17.	6 , __	6 × __ = 18
18.	3 , 4	3 × 4 = __
19.	7 , 5	__ × __ = 35
20.	__ , 4	__ × 4 = 24
21.	8 , __	40 = 8 × __
22.	__ , 4	__ × 4 = 28
23.	6 , 5	____________
24.	__ , 3	24 = ______
25.	5 , p	__ × p = 15
26.	n , 3	n × __ = 15
27.	n , p	__ × __ = 15

MULTIPLICATION FACTS

Exploration

It is important for you to learn how to multiply numbers quickly and correctly. This will help you to work exercises where an array can be made to picture the problem. Already you have learned some products by counting and by using addition. Now, you must learn how to multiply these small numbers and how to remember these facts well without having to look at arrays. In multiplying we think about two numbers and get a product. You already know many products. Think of some of them. What products do you know as you look at this array?

. . . .

. . . .

Did you use all or part of the array?

When you look at the part formed by a set of 2 rows and 8 columns, how many dots do you see?

When you look at the part of the array formed by a set of 2 rows and 7 columns, how many dots do you see?

When you look at the part of the array formed by a set of 2 rows and 1 column, how many dots do you see? Complete the following mathematical sentences.

2 × 9 = ____
2 × 8 = ____
2 × 7 = ____
2 × 6 = ____
2 × 5 = ____

2 × 4 = ____
2 × 3 = ____
2 × 2 = ____
2 × 1 = ____

What kinds of things come in groups of 2? Do they form an array?

Here is an array with 3 rows and 9 columns.

How many dots are in the array?

Cover one column. How many dots are in the array you see?

Cover one additional column. How many dots are in the array you see?

Using your 3 by 9 array, can you write all of the mathematical sentences as you did with the 2 by 9 array?

Name some things which come in groups of 3. Do they form an array?

We realize that we could not have an array without both rows and columns. However, as we work with the 3 by 9 array, we find that each time we cover a column, we are subtracting three elements from the total number of elements. Compare the information in the chart.

	3 × 9 = 27	You have a total of 27 elements.
Cover 1 column	3 × 8 = 24	27 - 3 = 24
Cover 2 columns	3 × 7 = 21	24 - 3 = 21
Cover 3 columns	3 × 6 = 18	21 - 3 = 18
Cover 4 columns	3 × 5 = 15	18 - 3 = 15
Cover 5 columns	3 × 4 = 12	15 - 3 = 12
Cover 6 columns	3 × 3 = 9	12 - 3 = 9
Cover 7 columns	3 × 2 = 6	9 - 3 = 6
Cover 8 columns	3 × 1 = 3	6 - 3 = 3
Cover 9 columns	3 × 0 = 0	3 - 3 = 0

By covering the last column you may imagine a 3 by 0 array (by pattern) or you may think of subtracting 3 elements. In either case there would be no (or zero) elements remaining. We know 3 - 3 = 0, so we may assume 3 x 0 = 0.

Here is an array. Describe it.

. . . .

. . . .

. . . .

. . . .

Using a piece of paper, cover the columns of the array one at a time. How many elements are there in the part you see? Begin with the whole array. Uncover the array one column at a time. Tell how many elements there are in the section you see each time.

Name some things which come in 4's and tell if they form an array? Using your 4 by 9 array, write your multiplication facts with factors of 4.

USING YOUR MULTIPLICATION CHART

Working Together

You have just completed the multiplication chart blank given you by your teacher. Using this chart, answer the following questions.

1. Why are all the products 0 in the first row and the first column?

2. Why are all the products in the second column the same as the numerals in the first column? Why are the products in the second row the same as the numerals across the top of the chart?

3. Read the products from the 9 column.
   Read the products from the 9 row.
   a. Read in order the digits in the ones' place of these products.
   b. How do the digits in the ones' place change?
   c. Read in order the digits in the tens' place of the products you read.
   d. How do the digits in the tens' place change?

4. What sum do you get if you add the numbers represented by the digits of each of the products in the 9 column?

5. Complete the following statements by using either "always" or "not always."

a. The product of two even numbers is ____________ an even number.

b. The product of two odd numbers is ____________ an odd number.

c. The product of an odd and an even number is ____________ an even number.

6. The product of any number and one (n × 1) is _____.

The product of any number and zero (n × 0) is _____.

Exercise Set 7

Make a chart with 4 columns as shown below. Study and complete the chart.

Numbers Operated On	Number which Results	Operation Used	Mathematical Sentence Showing Operation Used
Examples:			
4 , 2	6	Addition	4 + 2 = 6
7 , 4	3	Subtraction	7 - 4 = 3
4 , 3	12	Multiplication	4 x 3 = 12
1. 12 , 5	7	________	________
2. 6 , 4	24	________	________
3. 7 , 5	__	Multiplication	________
4. 4 , __	32	Multiplication	________
5. __ , __	7	Multiplication	________
6. 9 , __	16	________	________
7. 8 , 5	40	________	________
8. 5 , __	0	________	________
9. __ , __	15	Multiplication	________

PRACTICE IN MULTIPLICATION

Exercise Set 8

Complete the exercises 1 through 14 so they will form true mathematical sentences.

1. 5 × 7 =
2. 6 × 2 =
3. 4 × 6 =
4. 6 × 7 =
5. 8 × 3 =
6. 7 × 7 =
7. 7 × 8 =
8. 6 × 8 =
9. 8 × 9 =
10. 7 × 9 =
11. 6 × 6 =
12. 9 × 6 =
13. 8 × 8 =
14. 9 × 9 =

Write a mathematical sentence which goes with each problem. Solve it. Be sure you answer the question in a complete sentence.

15. There are 8 children planning to have a "cook out." Each child must bring 3 hot dogs. How many hot dogs will they have to cook, if they cook all the hot dogs?

16. One candy bar costs 7¢. How much would 8 candy bars cost?

17. A boy took 9 minutes to ride his bicycle to the store and home again (round trip). If he made 6 round trips in one day, how many minutes did he spend between his home and the store?

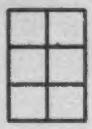
18. Jim swam one lap of the pool in 9 seconds. If he swam at the same rate, how long would it take him to swim 8 laps?

19. There are 6 pairs of gym socks in a box. If a salesman has 8 boxes of gym socks, how many pairs will he have?

20. There are 9 rows of chalk in a box, and there are 7 sticks of chalk in each row. How many sticks of chalk are in the box?

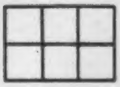
THE COMMUTATIVE PROPERTY OF MULTIPLICATION

A 3 by 2 array can be turned to form a 2 by 3 array.



3 by 2 Array

$3 \times 2 = 6$



2 by 3 Array

$2 \times 3 = 6$

This shows $2 \times 3 = 3 \times 2$.

A 78 by 65 array can be turned to form a 65 by 78 array. This shows

$$65 \times 78 = 78 \times 65$$

When we write 65×78 in place of 78×65, we are using the commutative property of multiplication. We can use the commutative property to reduce the number of multiplication facts we must remember.

Exercise Set 9

1. a. Look at the product in each of the following mathematical sentences.

   $3 \times 5 = m$ $\qquad$ $5 \times 3 = n$

   b. Does $m = n$?

   c. Using the two mathematical sentences in (a), make one sentence.

2. Decide if each of the following statements is true or false. Write _T_ if a statement is true. Write _F_ if a statement is false.

   a. $(4 + 3) = (3 + 4)$ $\qquad$ d. $(6 + 9) = (9 + 6)$

   b. $(7 \times 4) = (4 \times 7)$ $\qquad$ e. $(5 \times 8) = (8 \times 5)$

   c. $(12 - 5) = (5 - 12)$ $\qquad$ f. $(4 - 10) = (10 - 4)$

3. Look at your answers to problem (2) and answer these questions:

   a. Does addition have the commutative property?

   b. Does multiplication have the commutative property?

   c. Does subtraction have the commutative property?

COMPARING PRODUCTS

Exercise Set 10

Make each of the following mathematical sentences true by completing it with one of the symbols >, < , or = .

1. 1 × 12 ____ 4 × 3
2. 4 × 4 ____ 4 × 5
3. 2 × 6 ____ 4 × 3
4. 6 × 4 ____ 3 × 8
5. 5 × 5 ____ 4 × 8
6. 3 × 8 ____ 6 × 4
7. 7 × 4 ____ 9 × 3
8. 6 × 8 ____ 8 × 6
9. 9 × 4 ____ 6 × 6
10. 9 × 5 ____ 6 × 8
11. 7 × 5 ____ 7 × 3
12. 6 × 3 ____ 9 × 2
13. 9 × 6 ____ 8 × 7
14. 8 × 5 ____ 6 × 7
15. 7 × 8 ____ 6 × 9
16. 7 × 9 ____ 8 × 6
17. 6 × 9 ____ 7 × 8
18. 6 × 7 ____ 20 + 20
19. 8 × 9 ____ 8 × 8
20. 94 - 40 ____ 6 × 9
21. 9 × 9 ____ 8 × 9
22. 8 × 8 ____ 7 × 7
23. 8 × 7 ____ 10 × 6
24. 8 × n ____ 2 × n, where n ≠ 0
25. n × 6 ____ n × 8, where n = 0
26. 8 × n ____ 7 × n, where n ≠ 0
27. n × 4 ____ n + 4, where n > 1
28. n × 4 ____ n × 5, where n > 0

BRAINTWISTER: 29. 6 × 6 ____ 5 × 7 ____ 8 × 4

30. 8 × 5 ____ 7 × 6 ____ 5 × 9

FINDING UNKNOWN FACTORS

Exploration

Earlier we talked about an operation as thinking about two numbers and getting a third number. If we think about 4 and 6 and get 24, we are using multiplication. We can write the mathematical sentence:

$$4 \times 6 = 24$$

What are the numbers to be operated on? What is the result? What operation was used?

Look at these mathematical sentences.

$$\begin{array}{ccc} 3 \times 5 = p & & 3 \times n = 15 \\ & \text{and} & \\ 7 \times 2 = q & & m \times 2 = 14 \end{array}$$

How are these two sets of mathematical sentences different? How are the four sentences alike?

Do you multiply the two factors to find the unknown number in the sentence $3 \times 5 = p$? Do you multiply the two factors to find the unknown number in the sentence $7 \times 2 = q$?

Do you multiply the two factors to find the unknown number in the sentence $3 \times n = 15$? Do you know what number n represents? How do you know? Do you know what number m represents in the mathematical sentence $m \times 2 = 14$? How do you know?

In the sentence $5 \times r = 10$, to find r, ask yourself what factor times 5 is 10.

Let us try some others.

$$6 \times n = 12$$

$$t \times 3 = 15$$

What number does n represent? How do you know?

What number does t represent? How do you know?

Exercise Set 11

Find the unknown factor in the following sentences.

Example: $6 \times n = 24$

$n = 4$

1. $n \times 8 = 24$
2. $2 \times t = 16$
3. $p \times 4 = 16$
4. $n \times 1 = 16$
5. $p \times 6 = 48$
6. $8 \times p = 48$
7. $n \times 3 = 18$
8. $8 \times n = 0$

BRAINTWISTERS:

9. $n \times n = 25$
10. $n \times m = 23$
11. $(3 \times n) \times 2 = 24$
12. $785 \times n = 7{,}850$
13. $(n \times n) \times 4 = 36$
14. $(n \times n) \times n = 27$
15. $p \times p = p$
    (Here everything is missing!)

Exercise Set 12

Write a mathematical sentence which goes with each problem. Find the unknown factor. Draw an array if you need one. Be sure you answer the question in a complete sentence.

Example: Arrange 24 chairs in rows of 6 chairs each.
How many rows will there be?

$$n \times 6 = 24$$
$$n = 4$$

There will be 4 rows.

1. A class of 32 children was divided into groups of 8 for square dancing. How many dance squares were there?
2. Arrange 15 boys in 3 equal teams for a relay race. How many boys will there be on each team?
3. Mary is filling 3 Easter baskets. She has one dozen colored eggs. How many eggs can she put in each basket?
4. Bob arranged his collection of 28 butterflies in 7 rows. How many butterflies did he have in each row?
5. In a card game 32 cards were arranged face-up in rows with 4 cards in each row. How many rows were there?
6. The 36 children who direct traffic were divided into squads of 6. How many squads were there?
7. How many weeks are there in 35 days?

Exploration

Look at these mathematical sentences.

(a) $5 \times 2 = n$ (b) $5 \times n = 10$

What is n in the sentence $5 \times 2 = n$?

How do you know?

What is n in the sentence $5 \times n = 10$?

How do you know?

What operation is used to find a product?

In the sentence $5 \times n = 10$, n is an unknown factor. This operation of thinking about a number and one of its factors and finding an unknown factor is called division.

To divide we think what number times the known factor gives the product, or the known factor times what number gives the product.

Look at these mathematical sentences.

(a) $5 \times 8 = n$
(b) $5 \times r = 40$
(c) $p \times 3 = 21$

In the sentence $5 \times 8 = n$, 5 and 8 are factors, what is n called?

What operation is used to find n in the sentence $5 \times 8 = n$?

What number is n?

In the sentence $5 \times r = 40$, 5 is a known factor of the product 40. What is r called?

What operation is used to find r in the sentence $5 \times r = 40$?

What number is r ?

What did you think to find r in the sentence $5 \times r = 40$?

In the sentence $p \times 3 = 21$, 3 is a known factor of the product 21. What is p called?

What operation is used to find p in the sentence $p \times 3 = 21$?

What number is p ?

What did you think to find p in the sentence $p \times 3 = 21$?

Division is not always written as $5 \times n = 15$, or $n \times 5 = 15$ or $15 = n \times 5$. These sentences show multiplication.

This same relationship may be stated as a mathematical sentence which shows division. It may be written $n = 15 \div 5$. This sentence is read "n equals 15 divided by 5". n is the unknown factor; 15 is the product, and 5 is the known factor.

How may we describe division?

What must be given to find an unknown factor?

Can you find the unknown factor in these?

$7 \times m = 14$ $\quad$ $p \times 5 = 35$ $\quad$ $40 = 8 \times q$

How did you know these?

You thought of a multiplication fact in order to divide. If you can multiply, you can divide. This saves time.

For each multiplication fact you know, you can find some division facts. Let's try it. State some division facts you know because

6 × 8 = 48

5 × 9 = 45

8 × 9 = 72

Write each of these mathematical sentences as a mathematical sentence showing division.

a. 6 × n = 24

b. [illegible] × n = 12

c. n × n = 6

Summary

When we think of 6 and 2 and get 3, we are dividing. When we think of a number and one of its factors and get the other factor, we are dividing.

There are two ways to suggest division with mathematical sentences.

1. We can suggest division by a multiplication sentence with an unknown factor:

$$\underset{\text{(factor)}}{2} \quad \times \quad \underset{\text{(unknown factor)}}{n} \quad = \quad \underset{\text{(product)}}{6}$$

or

$$\underset{\text{(unknown factor)}}{n} \quad \times \quad \underset{\text{(factor)}}{2} \quad = \quad \underset{\text{(product)}}{6}$$

2. We also can suggest division by a mathematical sentence such as this:

$$6 \div 2 = n$$

We read this sentence

6 divided by 2 equals n .

If we think of 6 and 2 and get 3, we can write

$$6 \div 2 = 3.$$

Exercise Set 12

For each multiplication fact below, write two division facts:

Example: 8 × 6 = 48   48 ÷ 6 = 8
48 ÷ 8 = 6

1. 9 × 8 = 72

2. 7 × 9 = 63

3. 6 × 9 = 54

Rewrite each multiplication sentence as a division sentence. Find the unknown factor.

Example: 7 × n = 28   28 ÷ 7 = n
4 = n

4. 5 × n = 25
5. n × 8 = 24
6. 2 × t = 16
7. p × 4 = 16
8. n × 1 = 16
9. 9 × n = 72
10. p × 6 = 48
11. 8 × q = 48
12. n × 3 = 18
13. 8 × n = 0
14. n × 5 = 25
15. 8 × n = 64

Exercise Set 13

Mathematical Sentence	Operation Used	Unknown Number
Example: 5 × 8 = n	×	40
1. 5 × p = 40		
2. 15 - 13 = q		
3. 16 ÷ 8 = r		
4. 5 × n = 40		
5. 43 = n + 20		
6. p = 6 × 9		
7. 35 - n = 12		
8. t × 9 = 81		
9. 48 ÷ 6 = n		
10. 56 ÷ 8 = y		
11. 75 + 28 = n		
12. 49 ÷ 7 = n		
13. 49 - 7 = n		
14. 49 + 7 = n		

USING A MULTIPLICATION CHART TO DIVIDE

Working Together

In the multiplication chart, the numbers at the top and left side are factors. Numbers in the body of the table are products.

The multiplication chart can be very useful in finding division facts. To find n in $48 \div 6 = n$, for example, you may use either of the following ways.

(1) Think $6 \times n = 48$. Begin with row "6", and follow row "6" until you come to 48. Notice that 48 falls in column "8". Thus $6 \times 8 = 48$, so $48 \div 6 = 8$.

(2) Think $n \times 6 = 48$. Begin with column "6". Follow column "6" down until you come to 48. Notice that 48 falls in row "8". Thus $8 \times 6 = 48$, so $48 \div 6 = 8$.

Use your multiplication chart to find the unknown number in each of these division sentences.

(a) $42 \div 6 = n$

(b) $72 \div 8 = p$

(c) $54 \div 9 = r$

(d) $63 \div 7 = t$

Exercise Set 14

Use the multiplication chart to find the unknown number.

1. Complete the following division sentences using the chart.

   a. 72 ÷ 9 =

   b. 63 ÷ 7 =

   c. 45 ÷ 5 =

   d. 56 ÷ 8 =

   e. 81 ÷ 9 =

   f. 18 ÷ 2 =

   g. 32 ÷ 8 =

   h. 28 ÷ 7 =

   i. 64 ÷ 8 =

   j. 48 ÷ 6 =

The numeral 4 appears on the multiplication chart 3 times. A different mathematical sentence goes with 4 each time it appears. These mathematical sentences are:

$$1 \times 4 = 4$$

$$2 \times 2 = 4$$

$$4 \times 1 = 4$$

2. How many times does 36 appear on the chart?
   Write the mathematical sentences which go with 36.
   Be sure you state the number of rows first.

3. How many times does 24 appear on the chart?
   Write the mathematical sentences which go with 24.

4. How many times does 47 appear on the chart? Why?

RELATION OF MULTIPLICATION AND DIVISION

Multiplication will undo division. Think of 8, divide by 2, and then multiply by 2. The result is 8. The multiplication by 2 undid the division by 2.

$$(\underline{8} \div 2) \times 2 = \underline{8}$$

Division will undo multiplication. Think of 8, multiply by 2, and then divide by 2. The result is 8. The division by 2 undid the multiplication by 2.

$$(\underline{8} \times 2) \div 2 = \underline{8}$$

Exercise Set 15

1. Copy and complete the table below.

DO	UNDO
Examples: $2 \times 3 = 6$	$6 \div 3 = 2$
$12 \div 3 = 4$	$4 \times 3 = 12$
a. $1 + 4 = 5$	
b. $5 - 2 = 3$	
c. $n \times 6 = 24$	
d. $15 \div 5 = n$	
e. $5 \times n = 10$	

2. For each mathematical sentence, tell what operation is used to find n.

a. $2 \times n = 8$
b. $n \div 4 = 4$
c. $n = 8 - 3$
d. $28 = n \times 4$
e. $7 \times n = 42$
f. $36 \div 6 = n$
g. $48 - n = 8$
h. $27 = n \times 9$

Write a mathematical sentence which goes with each problem. Find the unknown factor. Be sure you answer the question in a complete sentence.

3. a. A checkerboard has 8 rows with 8 squares in each row. How many squares are there on a checkerboard?

   b. There are 64 squares on a checkerboard. There are 8 squares in a row. How many rows of squares are there?

4. a. An abacus shows 5 beads on each of 4 wires. How many beads are shown on the abacus?

   b. 20 beads are divided equally on each of the 4 wires of an abacus. How many beads are on each wire?

5. a. In the library, the card catalogue has 4 rows of drawers with 6 drawers in each row. How many drawers are in the card catalogue?

   b. The 24 drawers of the card catalogue in the library are divided into 4 rows. How many drawers are there in each row?

6. a. Bill had 30 heads of lettuce in his garden. They were evenly divided into 5 rows. How many heads of lettuce were there in each row?

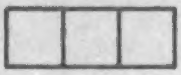
   b. Bill had 5 rows of lettuce in his garden. There were 6 heads of lettuce in each row. How many heads of lettuce did Bill have in his garden?

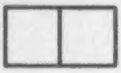
ONE OR ZERO AS A FACTOR

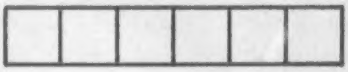
Exercise Set 16

1. What number is a member of every set of factors? Why?

2. Write a mathematical sentence suggested by each array shown below.

a. 

b. 

c. 

3. Find the unknown factor.

   a. $1 \times n = 9$

   b. $n \times 4 = 4$

   c. $1 \times n = 53$

   d. $72 \times n = 72$

   e. $923 \times n = 923$

4. Write the multiplication sentence belonging to a 1 by 287 array.

5. Write two of the factors of 2,877.

6. Complete the following sentences.

   Example: $2 \times 5 = 10$

   a. $1 \times 5 =$

   b. $0 \times 5 =$

   c. $3 \times 2 =$

   d. $3 \times 1 =$

   e. $3 \times 0 =$

   f. $1 \times 0 =$

   g. $0 \times 2 =$

   h. $2 \times 0 =$

7. a. What is $0 \times 9,728 = ?$

   b. What product do you always get when one of the factors is zero? Why?

   c. What numbers are factors of 0?

   d. Is 0 a factor of 0? Why?

8. How many arrays have 11 elements?

MULTIPLYING AND DIVIDING WHOLE NUMBERS

If two whole numbers are multiplied, we always get another whole number. We cannot always divide two whole numbers if we want a whole number answer.

There is not always a whole number to use as an unknown factor. For example,

$$7 \neq 3 \times n$$

and

$$4 \neq n \times 8$$

no matter what whole number n names.

It is always possible to multiply two whole numbers because there is always a third whole number to use as a product. This means the set of whole numbers is closed under multiplication.

It is not always possible to divide two whole numbers because there is not always a third whole number to use as an unknown factor. For example, there is no whole number n so that $10 \div 3 = n$. This means that the set of whole numbers is not closed under division.

Exercise Set 17

1. If possible, complete each mathematical sentence so it will be a true sentence. If there is no whole number to use as a result, answer "no".

   Example: $36 \div 9 = 4$

   a. $15 \div 15 =$

   b. $5 - 0 =$

   c. $8 \div 24 =$

   d. $7{,}285 \times 0 =$

   e. $15 - 25 =$

   f. $5 \div 5 =$

   g. $6 \times 7 =$

2. Complete the following statements by using either "always" or "not always." Give an example from exercise 1.

   a. Within the set of whole numbers, addition is __________ possible.

   b. Within the set of whole numbers, multiplication is __________ possible.

   c. Within the set of whole numbers, subtraction is __________ possible. Give an example.

   d. Within the set of whole numbers, division is __________ possible. Give an example.

3. Using what you discovered in 1 and 2 complete the following statements by using either "closed" or "not closed."

   a. Within the set of whole numbers, multiplication is _______.

   b. Within the set of whole numbers, division is __________.

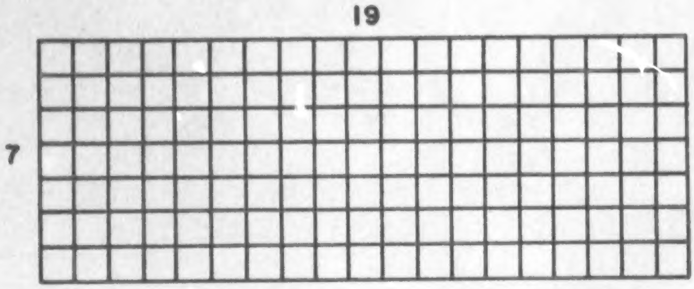
   c. Within the set of whole numbers, subtraction is __________.

   d. Within the set of whole numbers, addition is __________.

THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Working Together

Make an array like this one on a piece of paper you can fold.

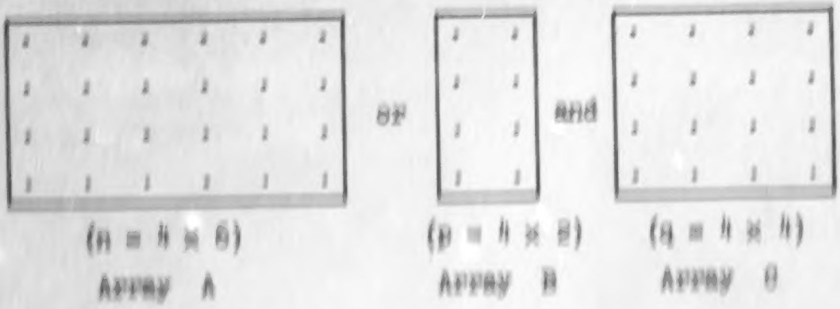


1. By folding the array you can separate the 19 columns of the array into two parts. This can be done in 18 different ways. Write a mathematical sentence for each fold you make.

   Example: 19 = 7 + 12

   19 = 3 + 16, etc.

2. Here is one array separated into two smaller arrays.



a. n is the number of dots in Array A. How many dots are there in Array A?

b. p is the number of dots in Array B. How many dots are there in Array B?

c. q is the number of dots in Array C. How many dots are there in Array C?

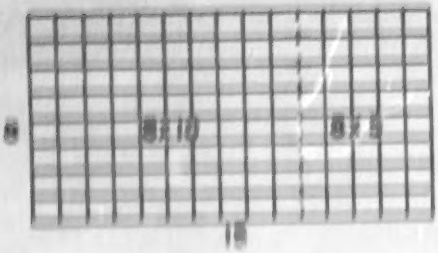
d. When you add the number of elements in Array B and Array C together, is the result the same as n?

e. Does $n = p + q$?

f. Does $24 = 8 + 16$?

g. Does $4 \times 6 = (4 \times 2) + (4 \times 4)$?

3.



8 × 15 = (8 × 10) + (8 × 5)

The dotted line shows a possible way to fold the array. The sentence below the picture shows the relation between the whole array and the two smaller arrays which the fold makes. The array can be folded in many other ways. Find 6 different ways of separating the array. Write the mathematical sentence for each separation.

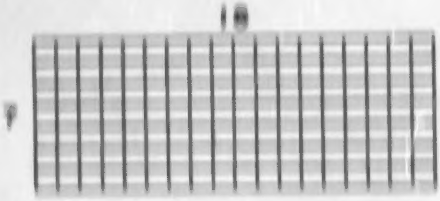
4. Make an array from squared paper to show the product of 9 × 13. Find 6 different ways of separating the array. Write the mathematical sentence for each separation.

5. Which of the sentences you wrote is the easiest one for you to use to find the product of 8 and 15?

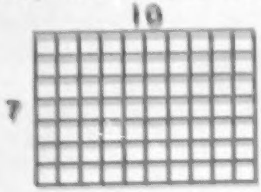
6. Which of the sentences you wrote is the easiest one for you to use to find the product of 9 and 13?

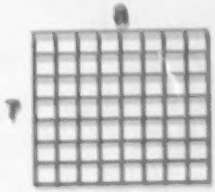
Summary

To find the product of 7 and 18, think of a 7 by 18 array:



Separate it into two arrays showing products you already know. For example,





Find the products separately and add them to get the total number of elements in the 7 by 18 array.

$$
\begin{aligned}
7 \times 18 &= 7 \times (10 + 8) \\
&= (7 \times 10) + (7 \times 8) \\
&= 70 + 56 \\
&= 126
\end{aligned}
$$

When we write $(7 \times 10) + (7 \times 8)$ in place of 7×18 we are using the distributive property of multiplication over addition.

USING THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Exercise Set 18

Rename the second factor of each of the following so it is convenient for you to multiply. Use the distributive property of multiplication over addition to find the product. Study the example before you begin.

Example: $4 \times 17 = 4 \times (10 + 7)$
$= (4 \times 10) + (4 \times 7)$
$= 40 + 28$
$= 68$

1. $3 \times 15 =$
2. $7 \times 12 =$
3. $5 \times 16 =$
4. $6 \times 19 =$
5. $8 \times 13 =$

Example: $7 \times 26 = 7 \times (20 + 6)$
$= (7 \times 10) + (7 \times 10) + (7 \times 6)$
$= 70 + 70 + 42$
$= 182$

6. $7 \times 26 = 7 \times (20 + 6) = (7 \times 20) + (\_ \times \_) = 140 + \_ = \_\_$
7. $5 \times 34 =$
8. $6 \times 28 =$
9. $9 \times 22 =$

Example:

$$\begin{aligned} 4 \times 153 &= 4 \times (100 + 50 + 3) \\ &= (4 \times 100) + (4 \times 50) + (4 \times 3) \\ &= 400 + 200 + 12 \\ &= 612 \end{aligned}$$

10. $3 \times 162 =$

11. $4 \times 147 =$

12. $5 \times 112 =$

13. $7 \times 120 =$

14. $6 \times 132 =$

15. $9 \times 111 =$

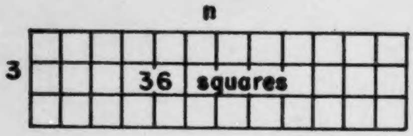
16. $3 \times 243 =$

17. $2 \times 361 =$

USING THE DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

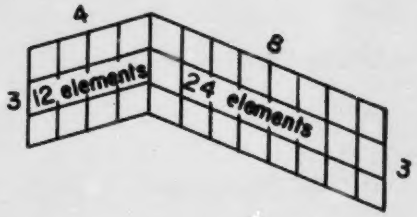
Working Together

1. Make an array of 36 squares. Use a piece of paper which may be folded. Make your array like the one below.



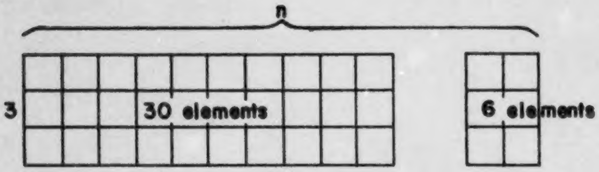
By folding the array, find 5 ways you can separate the 36 squares into two parts so that the number of squares in each part may be divided by 3. Write a mathematical sentence to show what you have done.

Example:



$36 = 12 + 24$

2. Suppose that you separated the array in Exercise 1 in this way.



$$36 = 30 + 6$$

We then can think:

$$36 \div 3 = n$$

$$36 \div 3 = (30 + 6) \div 3$$

$$36 \div 3 = (30 \div 3) + (6 \div 3)$$

$$36 \div 3 = 10 + 2$$

$$36 \div 3 = 12$$

3. Select another way in which you separated the 36-element array in Exercise 1. Show as we did above in Exercise 2 that $36 \div 3 = 12$ when using this other separation.

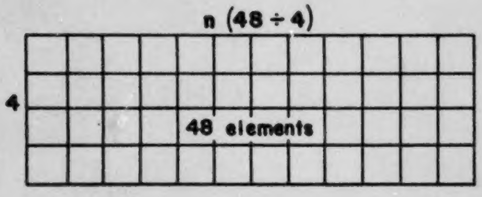
4. Do you think that one way of separating the array makes it easier to find $36 \div 3$ than other ways of separating the array? Why?

Summary

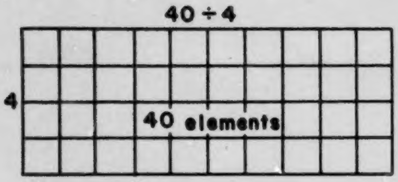
To find an unknown factor n as in

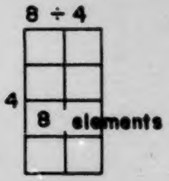
$$48 = 4 \times n$$

we can think of an array with 48 elements in 4 rows and n columns. If $48 = 4 \times n$, then $n = 48 \div 4$.



Think of separating this array into two arrays, so that the number of elements in each part can be divided easily by 4; for example:





Find the number of columns in each array. Add these numbers to find the number of columns in the 48 element array.

$$\begin{aligned} 48 \div 4 &= (40 + 8) \div 4 \\ &= (40 \div 4) + (8 + 4) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

When we write $(40 \div 4) + (8 \div 4)$ in place of $40 \div 4$, we are using the distributive property of division over addition.

PRACTICE WITH THE DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

Exercise Set 19

Using the distributive property of division over addition, find the unknown factor.

Example: $28 \div 2 = (20 + 8) \div 2$

$= (20 \div 2) + (8 \div 2)$

$= 10 + 4$

$= 14$

1. $66 \div 3 =$

2. $84 \div 4 =$

3. $62 \div 2 =$

4. $96 \div 3 =$

5. $88 \div 4 =$

6. $86 \div 2 =$

7. $69 \div 3 =$

BRAINTWISTER: Study the examples before trying exercises 8 through 12.

Example:

$$\begin{aligned} 284 \div 2 &= (200 + 80 + 4) \div 2 \\ &= (200 \div 2) + (80 \div 2) + (4 \div 2) \\ &= 100 + 40 + 2 \\ &= 142 \end{aligned}$$

Example:

$$\begin{aligned} 369 \div 3 &= (300 + 60 + 9) \div 3 \\ &= (300 \div 3) + (60 \div 3) + (9 \div 3) \\ &= 100 + 20 + 3 \\ &= 123 \end{aligned}$$

8. $468 \div 2 =$

9. $999 \div 9 =$

10. $693 \div 3 =$

11. $840 \div 8 =$

12. $75 \div 3 =$

THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

Working Together

1. Suppose you were told to multiply 2, 3, and 4.

$$2 \times 3 \times 4 = n$$

(a) Can you multiply the three numbers at the same time?

(b) Find the product when you multiply 2×3, and then multiply the result by 4.

(c) Find the product when you multiply 3 and 4 and then multiply the result by 2.

(d) Are the final products equal?

It is not possible to multiply more than two numbers at a time. If we have more than two numbers to multiply, we must group just two numbers.

In multiplying $2 \times 3 \times 4 = n$ we must multiply just two numbers at a time. To do this for $2 \times 3 \times 4 = n$ we can write

$$(2 \times 3) \times 4 = n .$$

The parentheses mean that we are grouping the 2 and 3 and we think of 2×3 as one number, 6 . Then the product is 6×4 or 24 . We could write.

$$2 \times (3 \times 4) = n .$$

This means we are grouping the 3 and 4. We think of this as one number 12. Then the product is 2×12 or 24. $(2 \times 3) \times 4$ and $2 \times (3 \times 4)$ are each names for the same number, 24. The way in which we grouped the numbers did not change the product. When we group $2 \times 3 \times 4$ as $(2 \times 3) \times 4$ or as $2 \times (3 \times 4)$, we are using the associative property of multiplication.

2. If we use the associative property of multiplication to write $n = 3 \times 2 \times 5$, we write

$$\begin{aligned} n &= (3 \times 2) \times 5 \\ &= 6 \times 5 \\ &= 30 \end{aligned}$$

or

$$\begin{aligned} n &= 3 \times (2 \times 5) \\ &= 3 \times 10 \\ &= 30 \end{aligned}$$

Find each product. Use the associative property of multiplication as was done above.

(a) $3 \times 2 \times 3$

(b) $4 \times 2 \times 3$

(c) $5 \times 2 \times 4$

3. (a) Tell how to do this operation: $(2 \times 4) \times 5$

(b) Tell how to do this operation: $2 \times (4 \times 5)$

(c) Why must we group two of the numbers in multiplying?

4. (a) What is the result of $(3 \times 5) \times 2$?

(b) What is the result of $3 \times (5 \times 2)$?

(c) Is $(3 \times 5) \times 2 = 3 \times (5 \times 2)$?

5. (a) Is $(3 \times 2) \times 4 = 3 \times (2 \times 4)$?

(b) Are $(3 \times 2) \times 4$ and $3 \times (2 \times 4)$ different names for the same number?

(c) In what way is $(3 \times 2) \times 4$ different from $3 \times (2 \times 4)$?

Summary

We can multiply the three factors 2, 3, and 4 in that order in either of two ways:

$$\begin{array}{ccc} (2 \times 3) \times 4 & & 2 \times (3 \times 4) \\ 6 \times 4 & \text{or} & 2 \times 12 \\ 24 & & 24 \end{array}$$

These two ways always give the same product. When we replace one way by the other, we are using the associative property of multiplication. Because both groups of factors give the same product, we can leave out parentheses and simply write

$$2 \times 3 \times 4 = 24.$$

USING THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

Exercise Set 20

Show how to use the associative property of multiplication to "prove" answers.

Example:

$$
\begin{aligned}
3 \times 2 \times 5 &= (3 \times 2) \times 5 \\
&= 6 \times 5 \\
&= 30
\end{aligned}
$$

or

$$
\begin{aligned}
3 \times 2 \times 5 &= 3 \times (2 \times 5) \\
&= 3 \times 10 \\
&= 30
\end{aligned}
$$

1. $2 \times 3 \times 4$
2. $4 \times 2 \times 6$
3. $3 \times 3 \times 2$
4. $4 \times 2 \times 3$
5. $2 \times 5 \times 4$
6. $3 \times 4 \times 2$

Use the associative property of multiplication to find these products.

Example:

$$
\begin{aligned}
3 \times 40 &= 3 \times (4 \times 10) \\
&= (3 \times 4) \times 10 \\
&= 12 \times 10 \\
&= 120
\end{aligned}
$$

7. 8×30
8. 9×40
9. 6×80
10. 5×40
11. 6×60
12. 8×40
13. 3×70
14. 4×20
15. 5×90
16. 4×40
17. 7×80
18. 3×60

USING THE PROPERTIES OF MULTIPLICATION

Exercise Set 21

1. Study the example below. Then copy the mathematical sentence filling in the unknown numbers. Write T if a statement is true and F if a statement is false.

Example: $(5 + 6) + 1 = 5 + (6 + 1)$

$11 + 1 = 5 + 7$

$12 = 12$ T

(a) $(2 \times 5) \times 3 = 2 \times (5 \times 3)$

$10 \times 3 = 2 \times \underline{\qquad}$

$30 = \underline{\qquad}$

(b) $(18 - 9) - 2 = 18 - (9 - 2)$

$\underline{\qquad} - 2 = 18 - \underline{\qquad}$

$\underline{\qquad} = \underline{\qquad}$

(c) $(25 \div 5) \div 5 = 25 \div (5 \div 5)$

$\underline{\qquad} \div 5 = 25 \div \underline{\qquad}$

$\underline{\qquad} = \underline{\qquad}$

2. Use the commutative and associative properties of multiplication to make the following products easier to find.

Example: $5 \times (2 \times 19)$ $\quad$ $5 \times (2 \times 19) = (5 \times 2) \times 19$

$= 10 \times 19$

$= 190$

(a) $(2 \times 27) \times 5 =$

(b) $(11 \times 4) \times 5 =$

(c) $[(5 \times 5) \times 7] \times (2 \times 2) =$

3. Use the commutative and associative properties of multiplication to find the following products.
   - (a) 20 × 40 =
   - (b) 30 × 30 =
   - (c) 30 × 40 =

4. Use the distributive property to show that 8 × 25 = 200.

5. In a chart showing all products from 0 × 0 through 9 × 9, there are 10 × 10 = 100 multiplication facts. If you know the commutative property of multiplication, what part of the chart do you really need? How many facts are given by this part of the chart?

DESCRIBING ARRANGEMENTS

Exploration

Joe, Sam, Ellen, and Janet were members of a stamp club. They each had some stamps to put in a stamp book.

Joe put 46 stamps on a page in his book.

He put 8 stamps in a row.

He made as many rows of 8 stamps as he could.

Here is a picture of the way Joe arranged his stamps.

```
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *
```

1. (a) Are there 8 stamps on each row?
   (b) How many rows of 8 stamps are there?
   (c) How many stamps are in the last row?

2. Tell if each of these sentences is true.
   (a) The 46 stamps are arranged in 5 rows of 8 stamps, with 6 stamps left over.
   (b) The set of 46 stamps is arranged in 5 sets of 8 and a set of 6.
   (c) $46 = (5 \times 8) + 6$.

3. Sam put 31 stamps on a page in his book.

He put 7 stamps in a row.

He made as many rows as 7 stamps as he could.

Use counters or draw a picture to show how Sam arranged his 31 stamps.

4. Tell in several ways how Sam arranged his stamps.

5. Complete this mathematical sentence to describe the way Sam arranged his stamps:

$$31 = (\underline{\ ?\ } \times 7) + \underline{\ ?\ }.$$

6. Ellen put 29 stamps on a page in her book.

She arranged as many of them as she could in 6 rows, with the same number of stamps in each row.

Use counters or draw a picture to show how Ellen arranged her stamps.

7. Complete each of these sentences to make it true.

   (a) The 29 stamps were arranged as 6 rows of _?_ stamps, with _?_ stamps left over.

   (b) The set of 29 stamps was arranged as 6 sets of _?_ and a set of _?_.

   (c) $29 = (6 \times \underline{\ ?\ }) + \underline{\ ?\ }$.

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Janet put 35 stamps on a page in her book.
She arranged as many of them as she could in 5 rows,
with the same number of stamps in each row.

8. Use counters or draw a picture to show how Janet arranged her stamps.

9. Complete each of these sentences to make it true.

   (a) The 35 stamps were arranged as 5 rows of ? stamps, with ? stamps left over.

   (b) The set of 35 stamps could be arranged as 5 sets of ? and a set having ? members.

   (c) 35 = (5 × ?) + ?

Use counters or drawings if needed to help you complete each of these sentences to make it true.

10. 23 = (? × 4) + ?.

11. 34 = (4 × ?) + ?.

12. 28 = (? × 7) + ?.

PARTITIONING SETS

Working Together

We can separate a set of things into subsets. When we do this, we may say that we partition the set.

We may try to partition a set so that each subset will have the same number of members.

Look carefully at the problems and the chart below. They may help us see two different ways to try to partition a set so that each subset will have the same number of members.

Tom had 23 pencils. He wanted to put them in 3 boxes, with the same number of pencils in each box. He wanted each box to have as many pencils as possible. How many pencils should he put in each box? How many extra pencils would he have?

Betty also had 23 pencils. She wanted to put them in small boxes, with 3 pencils in each box. She wanted to have as many of these boxes as she could. How many boxes of 3 pencils could she have? How many extra pencils would she have?

	Tom	Betty
Number of pencils in the set to be partitioned	23	23
Number of subsets, with the same number of pencils in each subset	3	n
Number of pencils in each subset	n	3
Number of pencils left over	r	r

We can use this mathematical sentence to help us think about partitioning Tom's set of pencils:

$$23 = (3 \times n) + r$$

Copy and finish this chart to show pairs of whole numbers that will make the mathematical sentence true.

n	r
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

We can use this mathematical sentence to help us think about partitioning Betty's set of pencils:

$$23 = (n \times 3) + r$$

Copy and finish this chart to show pairs of whole numbers that will make the mathematical sentence true.

n	r
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Your teacher will talk with you about some of the things we have learned from these charts. After this, rewrite each mathematical sentence so that it will be true, using the greatest whole number you can for n.

MAKING A RECORD OF OUR THINKING

Working Together

We have been using mathematical sentences like these when we partition sets of things in certain ways:

$$47 = (n \times 6) + r$$

$$47 = (6 \times n) + r$$

We try to make each sentence true using the greatest whole number we can for n. Knowing our multiplication facts will help us make true sentences in which we use the greatest whole number we can for n.

Look at this mathematical sentence: $47 = (n \times 6) + r$

We know that $7 \times 6 = 42$ and that $8 \times 6 = 48$. So, 7 is the largest whole number we can use for n: $47 = (7 \times 6) + r$

We now can think: $47 = 42 + r$, so we know that $r = 47 - 42$, or $r = 5$.

Now we can write: $47 = (7 \times 6) + 5$

We may call 5 the remainder.

Here are two ways to make a record of our thinking:

Form I

First write:	Then write:	Last write:
$6\overline{)47}$	$\begin{array}{r}7\\6\overline{)47}\\\underline{42}\end{array}$	$\begin{array}{r}7\\6\overline{)47}\\\underline{42}\\5\end{array}$

Form II

First write:	Then write:	Last write:
$6\overline{)47}$	$\begin{array}{r\|l}6\overline{)47}&\\\underline{42}&7\end{array}$	$\begin{array}{r\|l}6\overline{)47}&\\\underline{42}&7\\5&\end{array}$

Explain each of these ways to make a record of our thinking.

Now let us see how we can use each of these ways to make a record of our thinking when we partition sets.

We have 27 oranges. We want to make as many bags of 4 oranges as we can.

$$27 = (n \times 4) + r$$

Let us use Form I to record our thinking.

$$\begin{array}{r l} 6 & (n) \\ 4\overline{)27} & \\ \underline{24} & \\ 3 & (r) \end{array}$$

So, $27 = (6 \times 4) + 3$

We now know that with the ___ oranges we can make ___ bags of ___ oranges each and we will have ___ oranges left over.

Would we have found the same thing if we had used Form II instead of Form I to make a record of our thinking?

We have 27 oranges. We want to put them in 4 bags, with the same number of oranges in each bag.

$$27 = (4 \times n) + r$$

Let us use Form II to record our thinking.

$$\begin{array}{r|l l} 4\overline{)27} & & \\ \underline{24} & 6 & (n) \\ 3 & & (r) \end{array}$$

So, $27 = (4 \times 6) + 3$

We now know that with the ___ oranges we can make ___ bags of ___ oranges each and we will have ___ oranges left over.

Would we have found the same thing if we had used Form I instead of Form II in making a record of our thinking?

As you work with more problems you likely will find that you like one method better than the other. Learn to use the one you prefer.

Exercise Set 22

Copy and complete this chart.

Number in Whole Set	Number in Each Subset	Number of Subsets (n)	Remainder (r)	Mathematical Sentence
Example: 38	5	7	3	$38 = (7 \times 5) + 3$
1. 34	4			
2. 42	7			
3. 17	3			
4. 43	9			
5. 58	6			
6. 14	2			
7. 51	7			
8. 29	8			
9. 32	5			
10. 3	4			

Exercise Set 23

Copy and complete this chart.

Number in Whole Set	Number of Subsets	Number in Each Subset (n)	Remainder (r)	Mathematical Sentence
Example: 48	9	5	3	$48 = (9 \times 5) + 3$
1. 53	7			
2. 39	8			
3. 45	5			
4. 27	4			
5. 56	8			
6. 43	5			
7. 28	6			
8. 36	7			
9. 51	8			
10. 3	5			

Supplementary Exercise Set A

1. Complete the mathematical sentences below.

   a. $54 \div \underline{\qquad} = 6$

   b. $49 = 7 \times \underline{\qquad}$

   c. $8 + \underline{\qquad} = 17$

   d. $28 = 7 \times \underline{\qquad}$

   e. $\underline{\qquad} = 64 \div 8$

   f. $9 \times \underline{\qquad} = 72$

   g. $7 \times 8 = \underline{\qquad}$

   h. $12 \times 5 = \underline{\qquad}$

2. Copy and complete the chart below.

Mathematical Sentence	Unknown Number
a. $5 \times t = 30$	a. $t = \underline{\qquad}$
b. $6 + n = 15$	b. $n = \underline{\qquad}$
c. $12 \div p = 4$	c. $p = \underline{\qquad}$
d. $q \div 7 = 6$	d. $\underline{\qquad}$
e. $9 \times 9 = m$	e. $\underline{\qquad}$
f. $w \times 6 = 36$	f. $\underline{\qquad}$
g. $7 + 8 = y$	g. $\underline{\qquad}$
h. $54 \div 6 = v$	h. $\underline{\qquad}$

Supplementary Exercise Set B

1. If m has 3 as a factor and n has 2 as a factor, what factors are you sure $m \times n$ has?

2. a. How can you tell by looking at 7,485 that it has 5 as a factor?

   b. Can you replace the digit 5 by another digit and still have a number with 5 as a factor?

   c. Does 7,485 have 2 as a factor?

3. You are a "star" multiplier. You can multiply any two numbers. You are asked to find the number of seats in a very large auditorium. The seats form an array. How would you do it?

4. A marching band always forms an array when it marches. The leader likes to use many different formations. The band has 59 members in it. The leader is trying very hard to find one more member. WHY?

5. True or False? Every number with 6 as a factor has 3 as a factor also. Explain your answer.

6. If neither m nor n has 4 as a factor, then what do you know about the factors of $m \times n$?

7. How would you use the fact, $6 \times 9 = 54$, to find the unknown factor in $3 \times n = 54$?

8. You are told the multiplication fact 7 × 15 = 105.

   a. Find 7 × 16 from 7 × 15.

   b. Can you find 7 × 18 without first finding 7 × 17?

   c. Is there any correct way to fill the blank in

      7 × _____ = 101. WHY?

9. Multiply a number by 6. Divide the product by 3. The result is always an even number. Why is this true?

Supplementary Exercise Set C

1. Complete the following mathematical sentences.

   a. $(6 \times 6) \div 9 = \underline{\qquad}$

   b. $(6 \times 8) \div \underline{\qquad} = 6$

   c. $(4 \times \underline{\qquad}) \div 6 = 6$

   d. $(120 \div 12) \div 5 = \underline{\qquad}$

   e. $(63 \div 7) \times 6 = \underline{\qquad}$

   f. $(56 \div 8) \times \underline{\qquad} = 42$

2. Notice that $(24 \div 3) \div 2 = 8 \div 2 = 4$

   Also $24 \div 6 = 24 \div (3 \times 2) = 4$

   a. Is $36 \div 12 = (36 \div 4) \div 3$?

   b. Is $40 \div 8 = (40 \div 4) \div 2$?

   Can you see a general rule?

   Can you explain why it holds?

3. Use the trick we found in problem 2 to find

   Example: $72 \div 24$

$$
\begin{aligned}
72 \div 24 &= 72 \div (8 \times 3) \\
&= (72 \div 8) \div 3 \\
&= 9 \div 3 \\
&= 3.
\end{aligned}
$$

   a. $80 \div 16$

   b. $64 \div 16$

   c. $120 \div 20$

   d. $144 \div 24$

Notice that $(36 \div 6) \times 2 = 6 \times 2 = 12$.

Also $36 \div (6 \div 2) = 36 \div 3 = 12$.

Here's another: $(18 \div 2) \times 6 = 9 \times 6 = 54$.

Also $18 \times (6 \div 2) = 18 \times 3 = 54$.

What do you think the general rules are?

4. In each exercise, give a simpler way to get the same result.
   a. Multiply by 6. Then divide the result by 3.
   b. Divide by 10. Then multiply by 5.
   c. Divide by 4. Then divide by 3.
   d. Multiply by 2. Then multiply by 3.
   e. Multiply by 3. Then divide by 6.

5. In each exercise, write a pair of numerals which correctly fills the blank. (The same number may be used twice, but do not use the number 1.)
   a. $(5 \times \underline{\qquad}) \div \underline{\qquad} = 45$
   b. $(6 \div \underline{\qquad} \times \underline{\qquad} = 54$
   c. $(12 \div \underline{\qquad}) \div \underline{\qquad} = 1$
   d. $(3 \times \underline{\qquad}) \times \underline{\qquad} = 36$

6. Use what you did in exercise 4 to find:

   Example: $288 \div 24$

$$\begin{aligned} 288 \div 24 = (144 + 144) \div 24 &= (144 \times 2) \div 24 \\ &= 144 \div 12 \\ &= 12 \end{aligned}$$

   a. $144 \div 18$
   b. $144 \div 16$
   c. $108 \div 18$

7. Write the numerals which correctly fill the blanks. In any one exercise, fill all blanks with the SAME numeral.

a. $(\_\_\_\_ \div 8) \times 16 = 64$

b. $((\_\_\_\_ \div 8) \times 12) \div 6 = 10$

c. $(5 \times \_\_\_\_) \times \_\_\_\_ = 45$

d. $(4 \times \_\_\_\_) \div \_\_\_\_ = 8$

e. $(4 \times \_\_\_\_) + (6 \times \_\_\_\_) = 20$

Hint: Use the distributive property.

f. $(\_\_\_\_ \times 4) + (6 \times \_\_\_\_) = 80$

g. $(3 \times \_\_\_\_) + (\_\_\_\_ \times 7) = 20$

h. $(\_\_\_\_ \times 4) + (\_\_\_\_ \times 5) = 45$

i. $(\_\_\_\_ \times 5) + (2 \times \_\_\_\_) + (3 \times \_\_\_\_) = 70$

j. $6 \times (\_\_\_\_ + \_\_\_\_) = 48$

k. $(\_\_\_\_ \times \_\_\_\_) \div 9 = 4$

Supplementary Exercise Set D

1. The multiplication chart below has certain rows and columns out of place. Put the correct numbers in the left and upper margins.

	2	0	6	4
	0	0	0	0
	1	0	3	2
	3	0	9	6

2. $n \div 7$ is odd and is less than 10. The set of all factors of n has 3 members. What is n?

3. I'm thinking of a counting number. I will call it n.
   a. $3 \times n$ is an even number.
   b. $n \times 8 < 50$
   c. n has 3 as a factor.
   What is n?

4. Find a pattern in the list of products
   $1 \times 2,\ 2 \times 3,\ 3 \times 4,\ \cdots,\ 9 \times 10$

5. Find a pattern in the list of products
   $1 \times 3,\ 2 \times 4,\ 3 \times 5,\ \cdots,\ 8 \times 10.$

Practice Exercises

I. Find n in each of the following.

a) $28 + 7 = n$
b) $183 - 70 = n$
c) $56 + 21 = n$
d) $177 - n = 64$
e) $n + 28 = 60$
f) $173 + n = 184$
g) $68 + n = 159$
h) $137 - 34 = n$
i) $n - 55 = 102$
j) $44 + 72 = n$
k) $38 + n = 56$
l) $23 + 32 = n$
m) $49 - n = 25$
n) $42 + 65 = n$
o) $n - 81 = 144$
p) $88 + n = 159$
q) $24 + 56 = n$
r) $50 + 90 = n$
s) $n - 28 = 39$
t) $88 - 47 = n$

II. Use the associative property to find n in the following.

Example: $9 + 8 + 7 = n$ or $9 + 8 + 7 = n$
$(9 + 8) + 7 = n$ $9 + (8 + 7) = n$
$17 + 7 = n$ $9 + 15 = n$
$24 = n$ $24 = n$

a) $8 + 7 + 6 = n$
b) $26 + 9 + 14 = n$
c) $13 + 7 + 15 = n$
d) $24 + 16 + 11 = n$
e) $63 + 24 + 82 = n$
f) $19 + 16 + 14 = n$
g) $25 + 15 + 9 = n$
h) $46 + 17 + 3 = n$
i) $81 + 16 + 11 = n$
j) $23 + 17 + 43 = n$
k) $6 \times 3 \times 4 = n$
l) $5 \times 4 \times 2 = n$
m) $7 \times 2 \times 4 = n$
n) $8 \times 5 \times 2 = n$
o) $7 \times 3 \times 3 = n$
p) $4 \times 6 \times 2 = n$
q) $9 \times 3 \times 2 = n$
r) $7 \times 4 \times 2 = n$
s) $3 \times 5 \times 4 = n$
t) $6 \times 4 \times 8 = n$

III. Use distributive property to find s in the following.

Example:

$5 \times 35 = s$

$5 \times (30 + 5) = s$

$(5 \times 30) + (5 \times 5) = s$

$150 + 25 = s$

$s = 175$

$36 \div 6 = s$

$(30 + 6) \div 6 = s$

$(30 \div 6) + (6 \div 6) = s$

$5 + 1 = s$

$s = 6$

a) $7 \times 18 = s$

b) $9 \times 12 = s$

c) $8 \times 15 = s$

d) $25 \times 6 = s$

e) $43 \times 6 = s$

f) $16 \times 9 = s$

g) $12 \times 12 = s$

h) $37 \times 8 = s$

i) $7 \times 83 = s$

j) $93 \times 4 = s$

k) $28 \div 7 = s$

l) $42 \div 6 = s$

m) $66 \div 6 = s$

n) $54 \div 6 = s$

o) $72 \div 8 = s$

p) $36 \div 4 = s$

q) $32 \div 8 = s$

r) $24 \div 4 = s$

s) $36 \div 6 = s$

t) $18 \div 3 = s$

IV. Find the unknown number in each of the following:

a) $876 - 420 = t$

b) $678 + n = 989$

c) $445 + 458 = n$

d) $963 - 431 = a$

e) $86 + 94 + 77 = b$

f) $364 \div 4 = c$

g) $27 \times 8 = b$

h) $30 \div 5 = a$

i) $325 + 62 + 94 = t$

j) $35 \div n = 7$

k) $37 + 15 + 29 = r$

l) $36 \div 9 = s$

m) $309 + k = 466$

n) $9 \times r = 72$

o) $28 \div t = 7$

p) $a \div 6 = 7$

q) $118 \times 5 = m$

r) $54 \div p = 9$

s) $28 + 14 + 73 = r$

t) $s \div 8 = 8$

REVIEW

Set I

Part A

1. Write the symbols for Union or Intersection needed to fill the blanks between the set names. Example: a) ∩

Set A = {a, b, c, d, e, f}

Set B = {e, f, g, h, i, j}

Set C = {a, b, c, x, y, z}

a) A ______ B = {e, f}

b) B ______ C = { }

c) A ______ C = {a, b, c, d, e, f, x, y, z}

d) A ______ B = {a, b, c, d, e, f, g, h, i, j}

e) A ______ C = {a, b, c}

f) B ______ C = {a, b, c, e, f, g, h, i, j, x, y, z}

2. Write the word "even" or "odd" to make these sentences true. Example: a) even

a) The sum of any two even numbers is an ______ number.

b) The sum of any two odd numbers is an ______ number.

c) The product of any even and odd number is an ______ number.

d) The sum of any odd and even number is an ______ number.

e) The product of any two even numbers is an ______ number.

f) The product of any two odd numbers is an ______ number.

3. Write the word "even" or the word "odd" to make these sentences true.

a) 37 + 21 = n   n is an ________ number.
b) 13 × 4 = n   n is an ________ number.
c) 14 + 5 = n   n is an ________ number.
d) 472 × 2 = n   n is an ________ number.
e) 10 + 3 = n   n is an ________ number.
f) 22 + 14 = n   n is an ________ number.
g) 18 × 32 = n   n is an ________ number.

4. Apply the "undoing" idea to these operations. Some are done for you.

	Do	Undo
a)	16 - 6	(16 - 6) + 6
b)	15 + 9	____________
c)	4 × 3	(4 × 3) ÷ 3
d)	12 - n	____________
e)	8 × 2	____________
f)	32 ÷ 4	(32 ÷ 4) × 4
g)	14 × 2	____________
h)	25 + 5	____________
i)	17 - 7	____________
j)	27 ÷ 9	____________

5. Fill in the blanks.

a) 6 thousands + 26 hundreds + 2 tens + no ones =
8 thousands + ____ hundreds + 2 tens + no ones
The number is ________.

b) 2 thousands + 24 hundreds + 6 tens + 9 ones =

____ thousands + 4 hundreds + 6 tens + 9 ones

The number is ________.

c) 4 hundreds + 18 tens + 11 ones =

____ hundreds + ____ tens + 1 one

The number is ________.

6. Read carefully:

a) Mary said, "When I work my arithmetic problems I use only the numerals 0, 1, 2, 3, 4, and 5." What number base does Mary use?

b) Set B = {0, 1, 2, 3}

When Bill does his arithmetic homework he uses only the numerals in Set B. What number base does he use?

c) I have the place values of three, nine, twenty-seven, eighty-one, ... etc. What number base is being used?

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. On a spelling test of 50 words, Mary spelled 38 words correctly. How many words did Mary spell incorrectly?

2. There are six volleyballs in a carton. Our school buys four cartons of volleyballs. How many balls do they buy?

3. Ralph has a paper route. He delivers 82 papers each day. How many papers does he deliver on Monday through Friday?

4. Dick wanted a baseball cap that cost $1.25, a bat that cost $2.19, and a glove that cost $4.63. How much money would the three items cost?

5. Mr. Green drives 12 miles to work each morning. He drives 12 miles home each evening. How many miles does he drive each day? How many miles does he drive in a 5 day work week?

6. Janice is four feet three inches tall. Janice is how many inches tall?

REVIEW

Set II

Part A

1. Using the symbols >, <, or = make these true sentences.

   a) 7 ___ 6
   b) 0 ___ 1
   c) 5 × 5 ___ 125
   d) 3 + 199 ___ 200 + 10
   e) 264 ___ 268 - 2
   f) 120 ___ 189 - 9
   g) 6 + 8 ___ 15
   h) 2 + 6 ___ 2 + 6
   i) 8 - 1 ___ 3 × 3
   j) 5 - 2 + 3 ___ 6 + 2 - 2

2. Write these in base ten numerals.

   a) six thousand four hundred = ______
   b) four thousand one = ______
   c) seven hundred seven = ______
   d) one thousand ten = ______
   e) nine hundred thirteen = ______
   f) one hundred eight = ______

3. Write each of these sets in set notation.

   a) The set of even numbers greater than 40 but less than 55.
   b) The set of odd numbers less than 28 but greater than 16.
   c) The set of counting numbers between 45 and 51.
   d) The set of whole numbers less than 12.
   e) The set of the days of the week whose names begin with the letter "S".
   f) The set of children in this room who are two years old.

4. In the chart below, tell which property is illustrated by the number sentence at the left. Write the first letter of each word that names the property instead of writing the words. For example, write C P M for Commutative Property of Multiplication.

Number Sentence	Property Illustrated
a) $4 \times 3 = 3 \times 4$	
b) $6 + 18 = 18 + 6$	
c) $5 \times (2 \times 1) = (5 \times 2) \times 1$	
d) $8 \times 3 = 3 \times 8$	
e) $(9 + 1) + 6 = 9 + (1 + 6)$	
f) $(2 \times 6) \times 2 = 2 \times (6 \times 2)$	
g) $7 \times 13 = (7 \times 10) + (7 \times 3)$	
h) $35 \div 5 = (30 \div 5) + (5 \div 5)$	

5. Find what number is represented by y in each of these. Tell what operation is needed to find y. Use A for addition, M for multiplication, S for subtraction, and D for division. Example a) is done for you.

a) $6 + 9 = y$ $y = 15$ A
b) $y = 8 \times 5$ ______ ___
c) $19 - y = 14$ ______ ___
d) $y + 9 = 19$ ______ ___
e) $y = 8 + 4$ ______ ___
f) $4 \times 6 = y$ ______ ___
g) $24 \div 8 = y$ ______ ___
h) $7 \times y = 21$ ______ ___

6. A = {Joe, Jack, Tom}

B = {David, Donald}

What operation could you use to find the number of members in $A \cup B$?

Name the members of Set $A \cup B$.

7. W = {1, 2, 3, 4}

R = {0, 2, 5}

Could you use addition to find the number of members in $W \cup R$?

Name the members of the Set $W \cup R$.

8. $N \cup P$ = {a, b, c, d, e, f, g}

P = {a, b, c}

Could you use subtraction to find the number of members in set N?

Name the members of Set N.

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. On Saturday 60 people watched the Little League ball game. Forty-four people bought hot dogs. How many people did not buy hot dogs?

2. Mr. Brown took a trip. On Monday he drove 360 miles, Tuesday 419 miles, Wednesday 284 miles. How many miles did he drive altogether?

3. Carol bought 3 pairs of anklets. She paid 99 cents in all. How much did each pair of anklets cost?

4. Louis has 15 gum drops. He shares them equally with two of his friends. How many gum drops does each child receive?

5. Jerry's mother sent him to the store to buy a loaf of bread for 31 cents a loaf, a can of corn for 23 cents and two candy bars for 5 cents each. How much money should Jerry pay the clerk?
Jerry gave the clerk a one dollar bill. How much change should Jerry receive from the clerk?

REVIEW

Set III

Part A

1. Rename the following as base ten numerals:

a) 6 hundreds + 3 tens + 8 ones = ______

b) 5 hundreds + 13 tens + 8 ones = ______

c) 46 hundreds + 0 tens + 4 ones = ______

d) 4 thousands + 6 hundreds + 0 tens + 4 ones = ______

e) 4 thousands + 12 hundreds + 8 tens + 3 ones = ______

f) 16 hundreds + 0 tens + 11 ones = ______

2. In the chart below, tell which property is illustrated by the number sentence at the left. Write the first letter of each word that names the property instead of writing the words. For example, write A P M for Associative Property of Multiplication.

Number Sentence	Illustrated Property
a) $(4+5)+1=4+(5+1)$	
b) $8\times(1\times2)=(8\times1)\times2$	
c) $15+10=10+15$	
d) $4\times5=5\times4$	
e) $(3+12)+5=3+(12+5)$	
f) $12+42=42+12$	
g) $(4\times3)\times2=4\times(3\times2)$	
h) $28\div7=(14\div7)+(14\div7)$	
i) $12\times4=(10\times4)+(2\times4)$	

3. Perform the operation indicated. Tell whether the answer is an odd number or an even number.

Example:

a) 37 + 21 __58__ __even__
b) 13 × 4 _____ _____
c) 22 + 14 _____ _____
d) 18 × 32 _____ _____
e) 13 × 117 _____ _____
f) 14 + 5 _____ _____
g) 10 + 3 _____ _____
h) 472 × 3 _____ _____

4. After each statement write always true, sometimes true, or never true.

a) An even number added to an odd number gives an odd number. _______________

b) A whole number multiplied by a whole number gives a whole number. _______________

c) A whole number divided by a whole number is a whole number. _______________

d) When zero is added to a whole number the result is not a whole number. _______________

e) If the order of adding two whole numbers is changed the sum is changed. _______________

f) A whole number subtracted from a whole number gives a whole number. _______________

g) An even number multiplied by an even number gives an even number. _______________

h) An even number added to an even number gives an odd number. _______________

i) If the order of subtracting one whole number from another whole number is changed the unknown addend is changed. _______________

j) If the order of multiplying two whole numbers is changed the product is changed. _______________

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. When Bill went to the school party he counted the children there. After 14 children left he counted again and found 35 were still there. How many had Bill counted the first time?

2. The 851 children from Grant School came to watch a program that 183 fifth graders at Lee School gave. How many children were in the auditorium during the program?

3. There are 423 children in one school. Sixty-four children are in the fourth grade. How many children are not in fourth grade?

4. Miss Reed has 33 children in her class. Seventeen children are girls. How many boys are in the class?

5. John has 116 marbles. His younger brother has 77 marbles. How many marbles do the boys have altogether?

   How many more marbles does John have than his brother?

6. At a Little League candy sale, fifty boxes of candy were sold one day. Mark sold five boxes, Larry sold four boxes, and Tim sold eight boxes. How much candy did the three boys sell?

   How many boxes of candy did the other boys sell?

SUGGESTED ACTIVITIES

Group Projects

Problem Solution Progression

The first member of each team works one step of a given problem on the board, then returns to line. The next child works step two, etc. Any child detecting an error by a team mate may make corrections before his computation is made. The first team to complete a problem with the correct answer is winner.

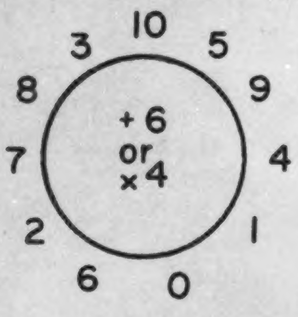
Example:

$$\begin{array}{r} 4368 \\ 9712 \\ 877 \\ \underline{+5645} \end{array} \qquad \begin{array}{r} 9712 \\ \underline{-4368} \end{array} \qquad \begin{array}{l} 315 \times 7 = \\ \\ 126 \div 6 = \end{array}$$

Game of Buzz as suggested in Chapter 4.

Clock Race

Draw a circle on the board. Write numerals from 0 through 10 around its face. Select a class member, indicate a starting point, and have him complete circling the face of the clock using the indicated process. Can be used for multiplication or addition.



Individual Projects

1. Make up new symbols for a number system. Prepare some examples using your new symbols (a chart, number line, etc.) to show and explain to the class.

2. Look in the encyclopedia to find number systems used by other countries. Make a chart that explains one (or more) for display.

3. Make your own "magic" square, circle or triangle. Put it on the board for the class to prove.

Puzzles

1. Each of the squares below is not quite magic. One number must be changed in each. Cross off the number to be changed and write the correct number.

a.

12	5	10
11	13	3
4	12	14

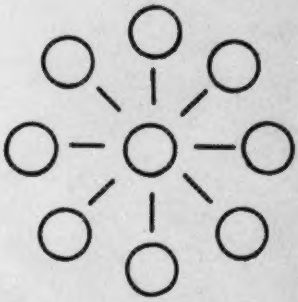
b.

10	31	15
20	9	4
3	16	14

c.

8	5	15
17	10	0
3	13	12

2. In each circle, place one of the numbers 11, 12, 13, 14, 15, 16, 17, 18, 19. The sum of the three numbers in each line must be 45.



3. **Find the Missing Digits . . . Don't give up too soon!** For each of these examples write only one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 in each blank to make the example correct.

a) $\begin{array}{r} 60\_\,\_ \\ -\ 1587 \\ \hline 4458 \end{array}$ b) $\begin{array}{r} 10463 \\ -\ \_\,\_\,\_\,\_ \\ \hline 2095 \end{array}$ c) $\begin{array}{r} 506 \\ \_ \\ 59 \\ +\ \_97 \\ \hline 1069 \end{array}$ d) $\begin{array}{r} 36\_ \\ 6\_0 \\ 13 \\ +\ \_64 \\ \hline 1606 \end{array}$

e) $\begin{array}{r} 96 \\ \times\ \_ \\ \hline 288 \end{array}$ f) $\begin{array}{r} \_\,\_\,\_ \\ \times\ 4 \\ \hline 3504 \end{array}$ g) $\begin{array}{r} 96 \\ \times\ \_ \\ \hline \_\,\_0 \end{array}$ h) $\begin{array}{r} \_2 \\ \times\ \_ \\ \hline 360 \end{array}$

Braintwisters

1. My brother is 5_ years old. On his next birthday he will be 10_. What number base am I using?

2. John's dog weighed 7_ pounds on Monday. When he was weighed again two weeks later he weighed 11_ pounds! The dog had gained 2_ pounds. What number base is John using?

3. The number of birds in a cage doubles every minute. The cage is full in half an hour. When was the cage half full?

4. An ant is climbing a post thirty feet high. It climbs three feet each day and slips down two feet every night. How long does it take the ant to reach the top of the post?

Chapter 5

SETS OF POINTS

LEARNING ABOUT SPACE

We are living in the space age. Man has already traveled in space and more space exploration will be done.

Suppose we were to plan a trip to Mars. Our space ship will have to follow a path which leads to Mars. Mars is moving all the time. To reach it, we must know its location in space, its speed in space, and its direction of travel.

The study of space and location is part of mathematics. This part of mathematics is called geometry. The things that we have been learning about the number system and about addition and subtraction belong to the part of mathematics called arithmetic.

To study geometry we need good imaginations. We make models and draw pictures to help us learn about things we cannot see. But our imaginations must help us too. Is your "imaginer" ready?

Geometry is not new. Thousands of years ago the Egyptians and Babylonians used ideas from geometry. It helped them to plan pyramids, lay out their fields, and study about the moon, stars, and planets.

The first geometry book was written about 2200 years ago. It had most of the ideas we still use in studying about space. However, many new ideas about geometry are still being discovered. Maybe you will be one to discover a new idea.

At first "geometry" meant "earth measure." But now geometry also uses ideas which do not involve measurements. In this unit called Sets of Points we are going to study some of these ideas.

We know that a "set" is a collection of things. Can you guess what a set of "points" would be? First you would need to know what a "point" is. In this unit we will learn about points, space, curves, line segments, rays, circles, polygons, and angles.

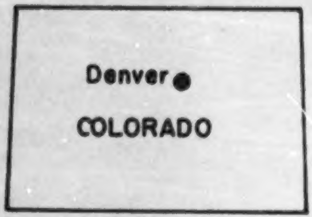
POINTS

What is a point? Is it the end of a sharp pencil? Is it the end of a needle? Is it the dot a pencil makes? Let's see what "point" means in geometry.

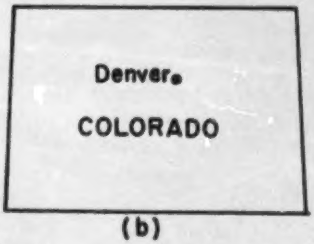
Working Together

1. Use your sharpest pencil to make a dot near the center of a sheet of paper. Now make a dot with a crayon. Next make a dot with a dull pencil. Do these dots look alike? In what way are they different? In what way are they the same?

2. Which of these maps of Colorado best shows the location of the state capital? Why?



(a)



(b)

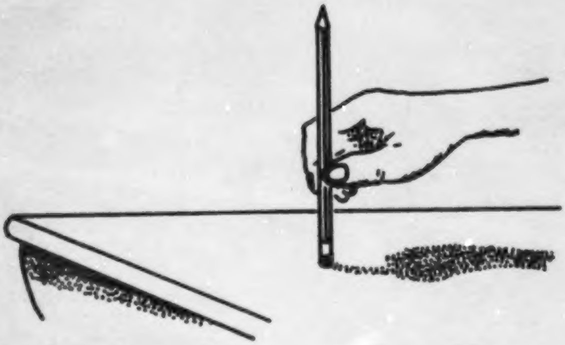
The dots you made in the first example and the dots in the maps of Colorado are attempts to show an exact location. The small dot marks the location more exactly. In geometry we often let a small dot represent a point. However, the dot is not the point any more than a picture of a cow is a cow.

A point in geometry means an exact location in space. Can you imagine something so small that you cannot see it? A point is so small it has no size at all.

Unless you have a microscope, you cannot see a germ. However, a germ covers many points as we think of them. If you were going to mark on a sheet of paper the locations covered by one germ, you would need a very sharp pencil. The dot made with the pencil would cover all these locations.

A very small dot is used to *represent* or *stand for* a point although a point is smaller than any dot which can be made.

3. Hold your pencil with its tip above your desk as in the drawing.



Could the sharpened tip of the pencil show a point? Move your pencil to another part of your desk. Does the tip now show a different point? Points do not move. They always stay in the same location.

In geometry we usually name pictures of points with capital letters like this:

A.   B·   C·

The points represented by A, B, and C can be called a "set of points." We will learn many interesting things about sets of points.

4. Describe a set of three points in your classroom.

5. Describe a set of two points in your classroom.

6. Describe a set of eight points in your classroom.

Exercise Set 1

1. Which of these is the best representation of a point?

A• B· C● D●

2. On your paper mark a set of five points using small dots. Label these pictures of points using the first five letters of the alphabet.

3. Write the letter of the best answer.
A dot made with a pencil covers
a) one point
b) one hundred points
c) several points
d) more points than can be counted

4. Which of these best describes a point?
a) a mark made with a pencil
b) a very very small dot
c) an exact location in space
d) a dot

SPACE

What is space? Is it air? Is it an empty place? Is it the distance from Earth to Mars? It is not any of these as we think of "space" in geometry.

Here are some examples of things which occupy sets of points in space:

The eraser on your pencil
The door to your classroom
Your little finger

Working Together

1. Now can you guess what "space" is? Which answer would you choose?

(a) Space is something hollow.

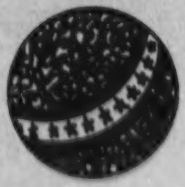
(b) Space is an object like a door or a finger.

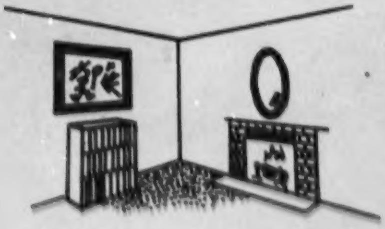
(c) Space is the set of all the exact locations everywhere.

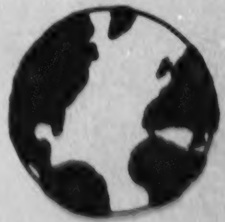
If you chose answer (c) you were correct. Space is the set of all points.

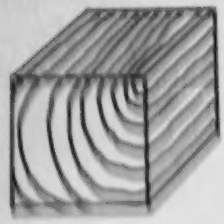
This means all exact locations everywhere. All the locations on the head of a pin, in your home, in your city and the sky above, in your country, in the world, and in the entire universe are points in space.

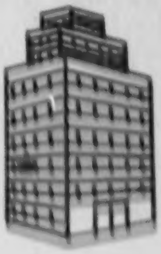
Space as we now picture it is probably very different from the idea you had. Any object you can think of covers or occupies lots of points of space. For example, a ball, a block of wood, a room, a building, the earth are all occupying parts of space.











2. Must a part of space be filled with air only?

3. Does a block of wood contain one point of space, a thousand points of space, or more points of space than can be counted?

4. Place a cup on a desk. It represents many points. Move the cup to some other place. Does it now represent the same set of points as before?

5. Place a block of wood on a desk. It represents many points. Move it to some other place. Does it now represent the same set of points as before?

Exercise Set 2

Write the letter of the right answer.

1. Which of these best tells what space is?

   a) Space is all empty places.

   b) Space is a set of points.

   c) Space is the set of all points.

   d) Space is the air around the earth.

2. In a truck load of grain, there are

   a) just as many points as there are grains.

   b) more points than there are grains.

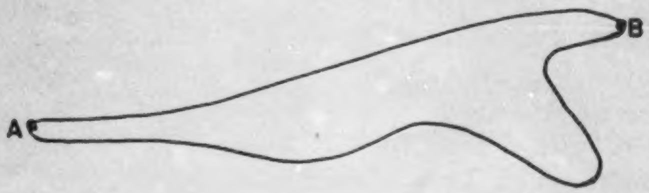
3. Which ones of these represent a part of space?

   a) A mark you make on your paper.
   b) The idea of truth.
   c) Your teacher.
   d) A tree.
   e) The idea of beauty.
   f) The crease in a piece of paper.

Working Together

1. Use two small bits of paper to mark two points on your desk. Trace with your finger to show ways you could go from one point to the other. How many different paths could you follow in going from one point to the other? Can you trace the most direct way to go from one point to the other?

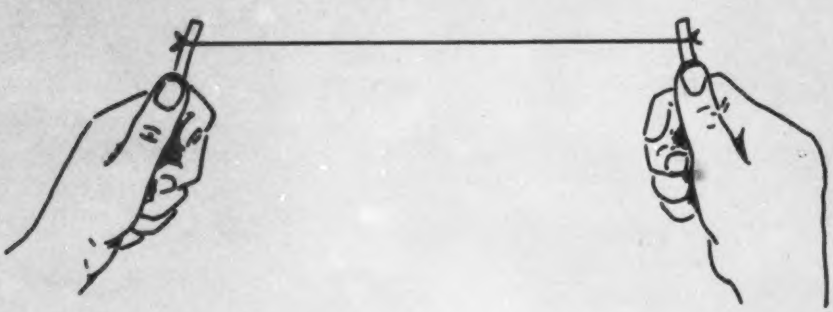
2. There are many ways of going from A to B. We see a picture of two ways.



Mark two points on a sheet of paper. Label them A and B. Show 5 ways of going from A to B on your paper. We do not have to stay on the paper. Think how you can go from A to B and touch the paper only at the dots.

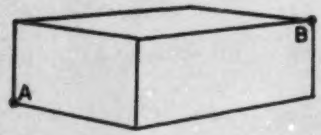
In going from one point to another, you have traced a curve with the tip of your finger or with your pencil. We think of a curve as a set of points. It is all the different locations your finger tip or pencil passes through in going from one point to another.

3. Let us think a little more about curves between two points. Suppose we use a piece of string tied to two pencils at the eraser ends to help us. We can let the erasers of the pencils mark two locations. Let us locate these points as far apart as the string will let us put our pencils. Does the string show the most direct path?



This direct path is a way of showing a special type of curve. We call it a line segment. Put dots on this string using chalk, pencil, or a pen. These dots mark points for us. We think of a line segment as a set of points. It is the set of all the points we have marked and all other points on our tightly stretched string. It also contains the two points represented by the erasers. We can show line segments in other ways.

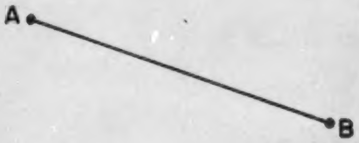
4. Mark on a block of wood the points A and B as shown in the picture.



Draw two curves on the block from point A to point B using two colors of crayon.

Did either curve you drew contain any line segments?

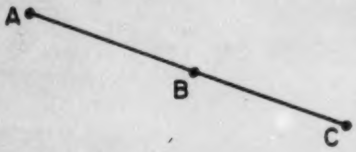
One excellent way to show a line segment is to draw a picture of one with a ruler and pencil. On your paper draw a segment connecting the two dots as shown in the figure below. We shall represent a line segment in this way.



We name this "line segment AB." A short way to write "the line segment AB" is $\overline{AB}$. $\overline{AB}$ means line segment AB. The line segment ends at points A and B. Therefore points A and B are called endpoints.

5. Think of the corner of your classroom as representing a point. What three things suggest line segments with this point as one endpoint?

6. Name all the line segments you see represented in this figure with endpoints in the set of points {A, B, C}.

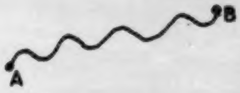


7. Mark a point on your paper. Call it point A. How many different line segments can you draw with A as an endpoint?

8. Give some examples of representation of line segments suggested by objects in your classroom.

9. Does your state have line segments as a part of its boundary?

10. Mark a point on your paper. Would you call your mark a line segment?

11. Mark something like this  on your paper. Is it a line segment?

12. Mark something on your paper which does not represent a line segment.

Exercise Set 3

1. Mark two points on your paper as is shown here.

   A   B

   Draw three different curves from A to B.

   Write the letter of the best answer.

   Each curve between these two points A and B goes through:

   a) one point

   b) three points

   c) many points

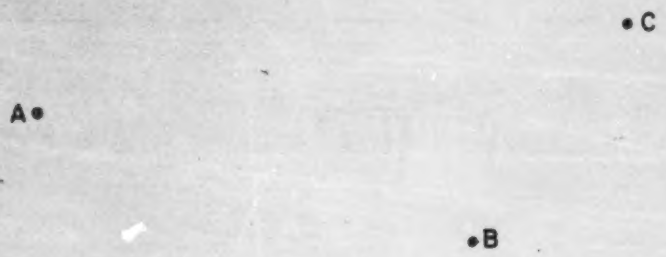
   d) more points than can be counted.

2. The set of points {A, B} is marked below. Copy this set on your paper. Draw as many line segments as possible having both endpoints in this set. How many are there?

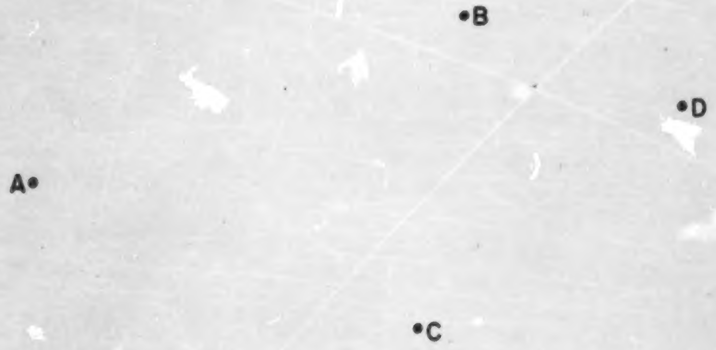
   A•

   •B

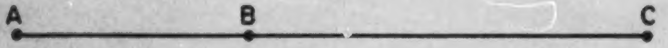
3. Copy the set of points {A, B, C} on your paper. Draw all the line segments having both endpoints in {A, B, C}. How many line segments are there?



4. Copy the set of points {A, B, C, D} on your paper. How many line segments can you draw, each having two of the points named as endpoints? Be sure to draw all the line segments. Name the line segments you drew.



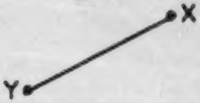
5. Name all the line segments you see in this figure. Both endpoints must be in the set, {A, B, C}.



6. Mark a point on your paper and label it A as shown below. Draw pictures of two line segments having A as an endpoint.

7. Mark a point on your paper and label it P. Draw pictures of three line segments having P as an endpoint.

8. Complete these statements on your sheet of paper.

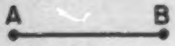


a) This is a picture of a __?__ __?__.

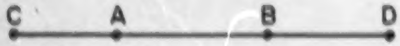
b) We write its name __?__.

Working Together

1. On your paper use a ruler to draw a line segment like $\overline{AB}$.

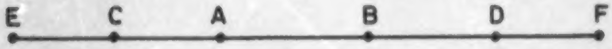


Draw a longer line segment which contains point A and point B by extending $\overline{AB}$ in both directions. Label the endpoints of this segment with the letters C and D. Does your drawing look something like this?



Is $\overline{AB}$ contained in $\overline{CD}$?

2. Draw an even longer line segment which contains points A and B by extending $\overline{CD}$ in both directions. Label the endpoints of this segment with the letters E and F. Your drawing might now look like this:

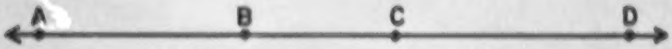


Is $\overline{AB}$ contained in $\overline{EF}$?

If you had a larger piece of paper and a longer ruler, you could draw a still longer line segment which would contain point A and point B. Think how this line segment would look if you were to draw longer and longer segments which contain points A and B.

Can you imagine how your drawing would look if it were extended without end? This is what we think of when we think of line. A line has no endpoints. It contains line segments of longer and longer length.

3. Below is a picture of a line.

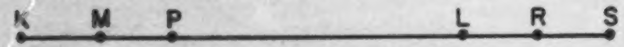


The arrows are used to show that it goes on and on in both directions without end. Only part of the line can be pictured on this page. We can call the line pictured, line AB. A short way of writing line AB is $\overleftrightarrow{AB}$.

Both A and B name points on the line. We know C and D name other points on the line. We could also call this line, line CD, or line AC, or line AD.

Line AB is the same as line BA. What other names can this line have? Use just the points named.

4. Here is a picture of $\overline{KS}$. M, P, L, and R, name other points of this line segment.



Copy $\overline{KS}$ and its labeled points.

a) Draw a line segment which pictures still more of the line PS and also includes line segment KS. Can you draw a complete picture of the line PL?

b) Draw a picture of a line AB. On your paper show a short way of writing line AB.

Remember that a line segment is a set of many points and a line also is a set of many points.

5. Follow these instructions carefully.

a) Mark a point on your paper and label it A. Draw one line through point A.

b) Now draw a different line through point A.

c) Next draw three more different lines through point A.

d) Mark one point different from point A on each of the lines you have drawn. Label the points with the letters B, C, D, E, F.

e) Can we draw more lines through point A?

f) Which is the correct ending to the sentence below.

Through point A we can draw:

one line.

more lines than you can count.

g) Describe the position of a line segment through A which is not on your sheet of paper.

6. On your paper mark two points, A and B.

•B

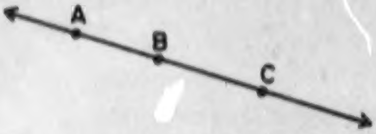
A•

a) How many line segments can you draw with endpoints A and B?

b) How many line segments can you draw which pass through both A and B?

c) How many lines are there that contain both A and B?

7. Below we have represented a line and three points of the line.



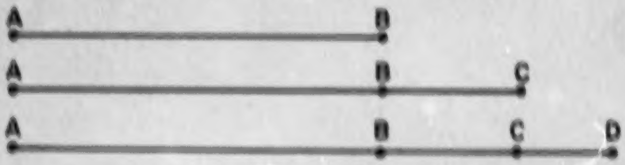
Shall we label this line $\overleftrightarrow{AB}$ or $\overleftrightarrow{AC}$?

In problem 6 we saw that with a ruler and a pencil only one line could be drawn through points A and B. From now on think of this statement as a fact; there is exactly one line that can be drawn through the two points A and B.

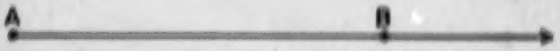
Working Together

1. Use a ruler to draw a line segment AB on your paper.

Now suppose we make longer and longer line segments but always keep A as one of the endpoints, as,



Then suppose we do not have a second endpoint, as in the picture below.

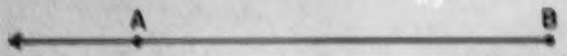


This gives us an idea for what is called a ray.

We can show a picture of only part of a ray on this page. We can name this ray, ray AB. Both A and B name points of the ray.

A ray has one endpoint. A is the name of the endpoint of the ray. A short way of writing ray AB is $\overrightarrow{AB}$. The endpoint is named first.

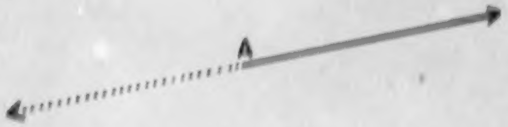
This is a picture of ray BA. What is its endpoint?



Ray BA is not the same as ray AB. Can you tell why? The endpoint of $\overrightarrow{BA}$ is B. What is the endpoint of $\overrightarrow{AB}$?

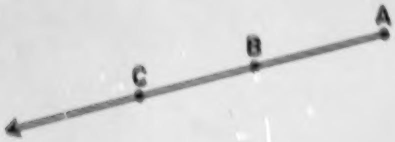
We can say that a ray is the union of the endpoint and all points on a line in one direction from this point.

For example, look at the line represented below and the point on it labeled A. One ray is represented by the solid part of the line. The other ray is represented by the dotted part of the line. The point A belongs to both rays represented and is called the endpoint of either ray.



A ray is always part of a line. A set of rays is nicely represented by a beam of light from a flashlight. Each starts at the flashlight and extends in one direction without end.

2. $\overrightarrow{AB}$ is represented below
   - a) Is $\overrightarrow{AC}$ another name for this ray?
   - b) Is $\overrightarrow{BC}$ another name for this ray?
   - c) Is the ray $\overrightarrow{BA}$ represented?
   - d) Is the ray $\overrightarrow{BC}$ represented?



Exercise Set 4

1. a) On your paper copy points F and E.

   Draw a picture of line segment FE.

   b) Mark two points on your paper and name them G and H. Draw a picture of the line through G and H.

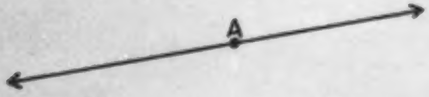
   c) Draw a picture of a ray. Name it $\overrightarrow{KL}$.

   d) Write the symbol for line segment FE; for line GH; for ray KL.

   e) What is the endpoint of $\overrightarrow{KL}$?

   f) Is $\overrightarrow{KL}$ the same ray as $\overrightarrow{LK}$? [illegible]y?

2. Let A be the name of a point of the line below. How many rays which are part of this line can have A as an endpoint?



3. Draw a picture of a line on your paper. Let A be a point of this line.

   a) Choose a point of the line different from A in one direction and label it B.

   b) Choose another point of the line in the other direction from A and label it C.

   c) Name two rays with endpoint A which are part of this line.

   d) Are there any more rays on this line which have A as an endpoint?

4. Label a point on your paper as A.

   a) Draw one ray with endpoint A.

   b) Draw another ray with endpoint A.

   c) Draw four more rays with endpoint A.

   d) How many rays could there be with A as endpoint?

5. On your paper draw a ray. Label it $\overrightarrow{AB}$. What is its endpoint? How many rays are there with endpoint A that contain point B?

6. On your paper draw a ray with endpoint A.
   - a) Choose a point on the ray different from A and label it B.
   - b) Is $\overline{AB}$ contained in $\overrightarrow{AB}$?
   - c) How many segments could there be on $\overrightarrow{AB}$ which have A as endpoint?

7. Mark two points on your paper and label them R and S.

•S

R•

   - a) How many lines can you draw which contain point R?
   - b) How many lines can you draw which contain point S?
   - c) How many lines can you draw which contain both R and S?

Which is the correct ending?

8. A line has
   - a) exactly 1 endpoint.
   - b) 2 endpoints.
   - c) no endpoints.

9. A ray has
   - a) exactly 1 endpoint.
   - b) 2 endpoints.
   - c) no endpoints.

10. A line segment has
    - a) exactly 1 endpoint.
    - b) 2 endpoints.
    - c) no endpoints.

Working Together

1. Can you find some flat surfaces in your classroom? Name as many as you can.

Do you know the geometric name for a set of points suggested by a flat surface? It is plane. Each flat surface you have named represents part of a plane.

2. Put your finger on a point on the top of your desk. Now put it on a different point. How many different points can you find on the flat top of your desk? How many points do you think there are in a plane?

3. Put your finger at a point above the top of your desk, then at a point below the top of your desk. Are there many points which are not in the plane represented by the top of your desk?

From now on we shall think of *a part of a plane* as a set of points in space. It is the kind of set suggested by all points on a flat table top, or on a wall, or on the floor. A piece of paper lying flat on your desk also suggests a part of a plane.

4. To get a better idea of what we mean by a plane, follow these directions.

a) See the picture of the figure below. Draw one like it near the center of a piece of paper.

b) Trace the figure with a (red) crayon. Color the part of the plane inside of the figure the same color. Give this colored part the name A. Is this colored region a picture of part of a plane?

c) Draw a bigger figure which encloses the colored region.

A. Color the larger figure and its inside (red) also. Name this new colored region, B.

Does the new colored region picture a part of the same plane as A?

Does colored region A or colored region B picture more of this plane?

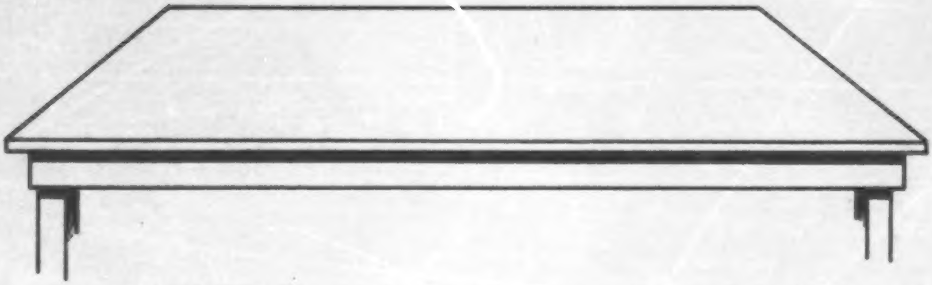
d) Draw a third figure which encloses the colored region B. Color this figure and its inside (red). Name this new colored region, C.

Does this new colored region picture a part of the same plane that A did?

Does colored region A or colored region B or colored region C picture more of this plane?

e) Can you draw a picture of the complete plane suggested by regions A, B, and C?

As we think of a line containing longer and longer segments so shall we think of a whole plane as containing larger and larger flat surfaces. Imagine your table top growing larger and larger on all sides. You would then have a table top upon which you could walk as far as you wished in any direction.



5. Does the set of points represented by the table top move when the table is moved?

6. Name some other objects which represent parts of planes.

7. Is there more than one plane in space?

We shall often use a sheet of paper placed on a flat table or desk top to represent a part of a plane. The table top itself may be thought of as containing even more points of this same plane.

Exercise Set 5

1. Does a plane as we shall think of it contain one point or more points than can be counted?

2. Take a sheet of paper. Think of it as part of a plane. Is it possible to draw more than one line in this plane?

   If so, draw two lines.
   Now draw three more lines.
   Draw ten more lines.

3. Does a plane contain one line, two lines, or more lines than can be counted?

4. Think of the top of your desk as a part of a plane. Describe the location of a point not on this plane. Describe the location of a line not on this plane.

5. Does a plane contain all points of space?

6. Does a plane contain some lines but not all lines in space?

7. Take a sheet of paper. Think of it as part of a plane. Describe a line which is not on this plane. Draw a ray which is on the plane. Describe a ray which is not on this plane.

8. If the endpoint of a ray is not on a certain plane, is the ray on that plane?

9. If the endpoint of a ray is on a certain plane, then must the ray be on that plane?

10. If two points of a ray are on the plane, then must the ray be on the plane?

LINES AND PLANES

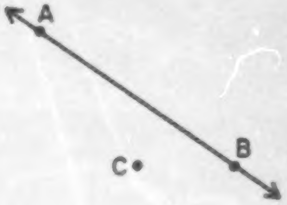
Working Together

Let us think about a line and a plane. Suppose the line has two of its points in the plane. For example, look at the points A and B represented on this page. The page suggests part of a plane which contains A and B.

1. Answer these questions carefully.

a) How many lines contain both points A and B?

b) Are all of the points of $\overleftrightarrow{AB}$ contained in a plane which contains A and B?



c) Think of a third point in the plane and label it C. Draw line CA. Is $\overleftrightarrow{CA}$ in the plane?

d) Draw line CB. Is $\overleftrightarrow{CB}$ in the plane?

Suppose we have two points, A and B. Suppose we have a plane called E. If point A is in plane E and point B is in plane E, then the entire line AB is in plane E.

2. Which is the correct ending:

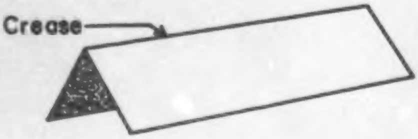
A line with two of its points contained in a plane

a) has some, but not all, of its points contained in that plane.

b) has all of its points contained in that plane.

3. Can there be more than one plane containing two points? If there is more than one plane, give an example. Remember there is just one line containing these two points.

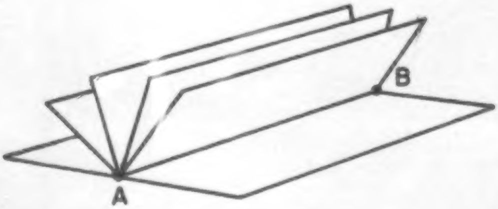
4. Fold a piece of paper in half. We think of the crease as a line segment. Stand the folded paper on your desk so that the crease does not touch it. The paper makes a tent.



Does this suggest parts of two planes which contain the line segment represented by the crease? If so, show them.

5. Give an example of two points and three planes passing through them.

6. Open a thin book so that you see the pages as in the figure below. The spine of the book suggests a segment. Name it $\overline{AB}$.



a) What does each page suggest?

b) Does each page pass through the spine of the book?

7. Choose two points in space. Now how many planes do you think contain the same two points?

8. Choose a line segment in space. How many planes do you think contain this line segment?

9. Choose a line in space. How many planes do you think contain this line?

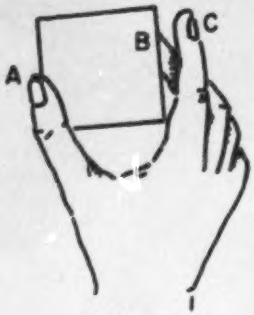
10. Which is the correct ending?

a) Two points in space are contained in
   1) only one plane.
   2) many, many planes, but we could count them.
   3) more planes than can be counted.

b) A line segment is contained in
   1) only one plane.
   2) many, many planes, but we could count them.
   3) more planes than can be counted.

c) A line is contained in
   1) only one plane.
   2) many, many planes, but we could count them.
   3) more planes than can be counted.

11. We are now going to make a very important observation. Hold a piece of cardboard at the middle of the side edges by the thumb and middle finger as shown below.



Without moving the thumb and middle finger we are able to use our other hand to rotate the cardboard to many positions showing many planes through $\overleftrightarrow{AB}$.

a) Now rotate the cardboard until it touches the tip of your index finger. Think of the tip of this finger as point C.

The card now represents a plane passing through the points A, B, and C.

Does there seem to be another plane passing through points A, B, and C?

This suggests that through three points not on a line there passes just one plane. We shall think of this as a geometric fact.

b) Think of a door representing part of a plane and the hinges representing two points. As the door swings open, does it suggest many planes through these points?

Now hold a finger tip against the door. Your finger tip represents a third point which holds the door open in one position.

This again suggests that through three points not on a line there passes just one plane.

12. Review

a) A plane contains more points and more lines than can be counted.

b) If two points of a line are contained in a plane, the whole line is contained in the plane.

c) Through two points there passes more planes than can be counted.

d) Through three points not on a line there passes one and only one plane.

INTERSECTIONS OF LINES AND PLANES

Working Together

Do you recall what we mean by the intersection of two sets?

1. Set A = {2, 3, 5, 9} Set B = {1, 2, 3, 4, 9}

The intersection of A and B is { }.

2. Set R = {M, A, R, Y} Set S = {C, A, N, D, Y}

The intersection of R and S is { }.

The union of R and S is { }.

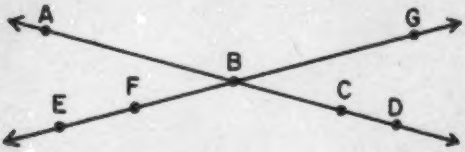
3. Set E = {5, 6, 7, 8} Set F = {9, 10, 11, 12}

The intersection of E and F is { }.

The union of E and F is { }.

You know that a line is a set of points and a plane is a set of points. Let us find the intersection of two lines.

Look at the points named on the lines in the picture below.



4. What points of $\overleftrightarrow{AC}$ are labeled? Of $\overleftrightarrow{EG}$?

5. Is there any point that is on both lines?

6. What is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{EG}$?

What is the intersection of $\overrightarrow{BA}$ and $\overrightarrow{BG}$?

What is the intersection of $\overrightarrow{FE}$ and $\overrightarrow{BG}$?

7. How many points are in the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$?

8. Can you hold two pencils to represent lines so that no point is on both lines? Can you do this in more than one way?

9. Use a card to represent a plane and a pencil to represent a line. Can you hold them to make their intersection

a) one point?

b) no points?

c) just two points?

d) many points?

10. Use two cards to represent two planes. Can you hold them so the intersection of the planes they represent is

a) just one point?

b) just two points?

c) more than two points?

d) no points?

11. Which of these pictures show that:

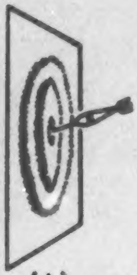
a) the intersection of a line and a plane is one point?

b) the intersection of a line and a plane is a line?

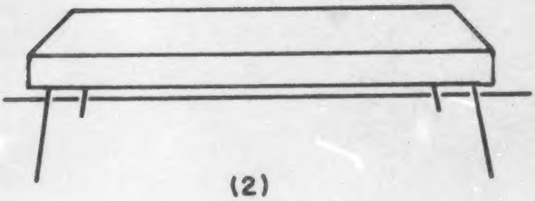
c) the intersection of two planes is a line?

d) the intersection of two lines is a point?

e) the intersection of two lines is the empty set?



(1)



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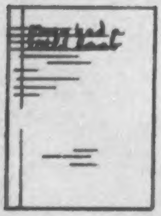
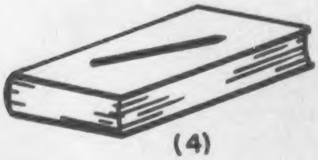
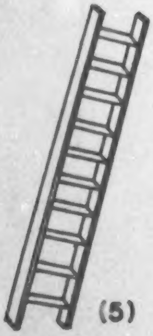


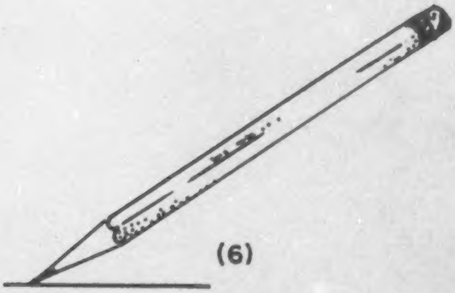
(3)



(4)



(5)



(6)

Exercise Set 6

1. Mark a point on your paper and label it A. Draw two different lines through A.
   What is the intersection of the two lines?

2. Mark two different points on your paper and call them B and C. Can you draw two different lines, both through B and C? Can you draw one line through both B and C?

3. What word will make this a true sentence?
   If two different lines in a plane cross, their intersection is _____?_____ point.

4. Can you draw a picture to represent two lines whose intersection seems to be the empty set? If so, draw it.

5. Look at the walls, floor, and ceiling of your classroom. Which represent pairs of planes which cross?
   a) the side wall and front wall
   b) the floor and ceiling
   c) the back wall and front wall
   d) the front wall and ceiling

6. Which of the walls, floor, and ceiling represent planes which do not cross?
   a) the floor and side wall
   b) the floor and ceiling
   c) the back wall and front wall
   d) the front wall and ceiling

7. Imagine you have folded a sheet of paper and opened it to form a tent as we did before. Does the folded sheet suggest two intersecting planes? What is the intersection in this case?

8. Complete this sentence. Two intersecting planes in space intersect in a ______?______.

9. If three different points of a line are in a plane, what can you say about the line and the plane?

10. Review

    We have learned the following facts.

    a) If two different lines in a plane cross, their intersection is one point.

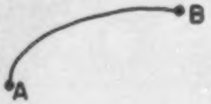
    b) If two different planes in space cross, their intersection is one line.

    c) If a line and a plane cross, the intersection is either one point or the entire line.

SIMPLE CLOSED CURVES

Working Together

In the section on curves, we drew a path from a point A to a point B. We called the set of points the tip of the pencil passed through a curve.



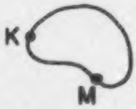
1. Mark two points on your paper and name them T and R. Draw on your paper a picture of a curve from T to R.

2. Mark two points F and H. Draw $\overline{FH}$. We also call $\overline{FH}$ a curve.

3. a) Mark a point K. Draw a curve that starts at K and comes back to K along a different path. Could you draw the curve using line segments?

Since your curve begins and ends at the same point, it is a closed curve.

b) Mark a different point on the curve that contains K and call it M. Can you start at M and trace the curve and come back to M? Did you trace every point of the curve?



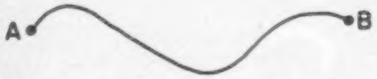
4. Mark a point A and a point B. Draw a curve that begins at A and passes through B and then comes back to A without crossing itself.

Your curve through A and B is called a simple closed curve. It starts at one point and comes back to this point without intersecting itself. All the points of a simple closed curve are in the same plane.

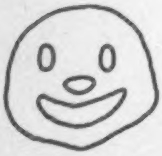
5. Mark a point C and a point D. Draw a curve that starts at C and passes through D twice and then comes back to C.

Your curve through C and D is not a simple closed curve because it intersects itself at D.

6. The curve below does not intersect itself. Why is it not a simple closed curve?



7. Is the figure below a simple closed curve? Why?

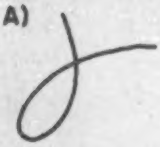


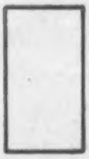
8. A frame around a picture suggests a simple closed curve. Name some other things which suggest simple closed curves.

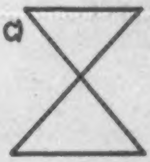
Exercise Set 7

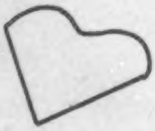
1. Draw a simple closed curve on your paper. Draw it with a blue crayon. Color red the part of the plane inside the curve. Color green the part of the plane outside the curve. (Can you color all of this plane?)

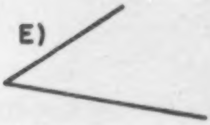
2. Tell which of the following are pictures of simple closed curves.

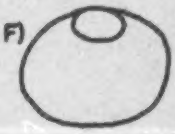
A)


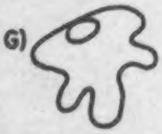
B)


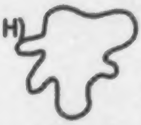
C)


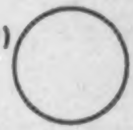
D)


E)


F)


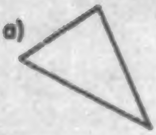
G)


H)


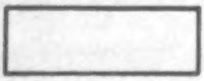
I)


3. Which of the pictures are not simple closed curves?

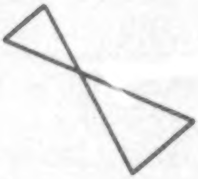
a)



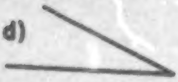
b)



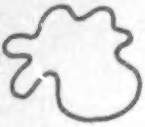
c)



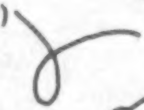
d)



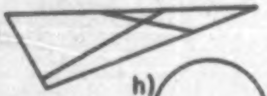
e)



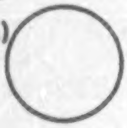
f)



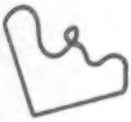
g)



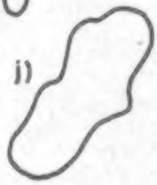
h)



i)



j)

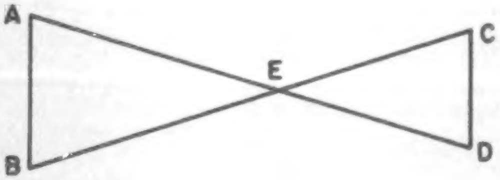


k) hello

4. Do the boundaries of most states on a map of the United States represent simple closed curves?

5. Name one state whose boundary on the map of the United States does not represent a simple closed curve. Name another such state.

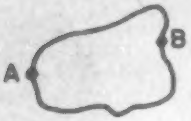
6. Is the figure below a simple closed curve?

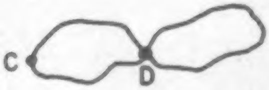


Is it the union of simple closed curves? How many simple closed curves?

7. Look at the curves in Ex. 3. Give other names for some of the simple closed curves.

8. Did your curves for Ex. 4 and 5 of page 271 look something like these?





POLYGONS

Working Together

1. Draw a simple closed curve which is the union of

a) three line segments. What is a name for your figure?

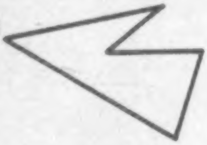
b) four line segments. What is a name for your figure?

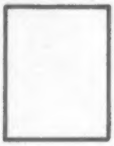
c) five line segments. What is a name for your figure?

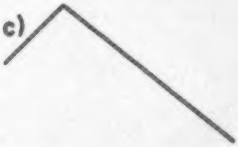
2. Can you draw a simple closed curve with two line segments? Why?

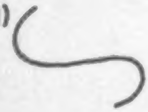
A simple closed curve which is the union of line segments is called a polygon.

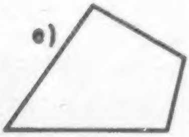
3. Which of these are pictures of simple closed curves? Which are pictures of polygons?

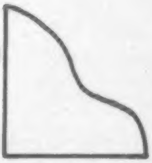
a)


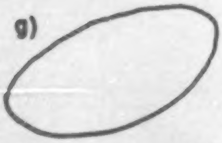
b)


c)


d)


e)


f)


g)


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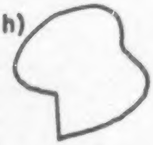
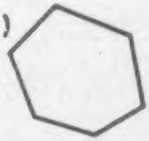
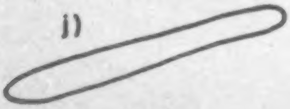
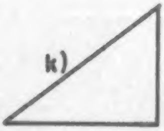
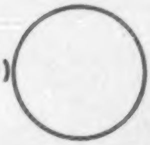


i)


j)


k)


l)


4. Describe some line segments in your classroom which form polygons.

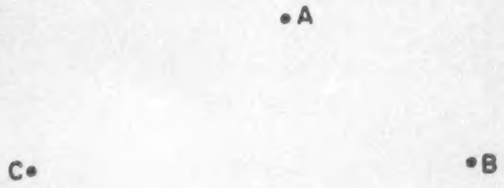
A polygon which is the union of three line segments is a triangle.

A polygon which is the union of four line segments is a quadrilateral.

5. Which of the pictures in Ex. 3 are pictures of triangles?

6. Which of the pictures in Ex. 3 are pictures of quadrilaterals?

7. Mark three points on your paper like these. Draw $\overline{AB}$, $\overline{CB}$, $\overline{AC}$.



a) Does the figure represent a polygon?

b) Does the figure represent a triangle?

c) What words should complete this sentence?

A triangle is made up of ____?____ line segments and these line segments have ____?____ different endpoints.

We usually label the endpoints of the segments in a triangle by capital letters, such as, A, B, C, and use the name or symbol $\triangle ABC$. Another equally good name is $\triangle BAC$.

8. Can you give another name for the triangle?

Exercise Set 8

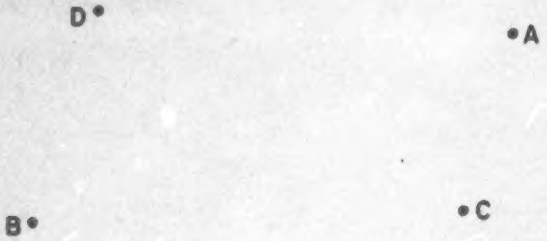
1. a) Draw a picture of a simple closed curve that is the union of three line segments.
    b) Label the endpoints of the three line segments.
    c) How many different endpoints are there?
    d) What are two names for this kind of simple closed curve?

2. a) Draw a picture of a simple closed curve that is the union of four line segments.
    b) Label the endpoints of the four line segments. How many endpoints are there?
    c) What are two names for this kind of simple closed curve?

3. a) Draw a picture of a simple closed curve that is the union of five line segments.
    b) What is a name for this kind of simple closed curve?

4. Locate on your paper points like these below.

Draw $\overline{AD}$, $\overline{DB}$, $\overline{CB}$, and $\overline{AC}$.



5. Which of these are names for this figure?

a) simple closed curve

b) polygon

c) triangle

d) quadrilateral

6. Draw $\overline{AB}$ in your drawing for Ex. 4. How many triangles do you see? Name them.

7. Now draw $\overline{CD}$ in the same figure. Mark the intersection of $\overline{AB}$ and $\overline{DC}$. Label it E. How many triangles do you see now? Name them.

8. Draw a set of points which is the union of three line segments. Draw a closed curve which is the union of three line segments. Can these drawings be different? Can these drawings be the same?

9. Could a polygon have exactly 8,999 sides?

10. Could a polygon be the union of two line segments and part of the letter O?

11. Is the letter O a polygon?

CIRCLES

Working Together

1. Mark a point on your paper and label it A. Mark another point two inches from point A.

2. Mark many other points which are two inches from the point marked A.

3. Mark some more points which are two inches from the point marked A. Be sure they are in all directions from point A.

4. Do the points you marked suggest a simple closed curve? (Do not use the point marked A.) If not, mark some more points which are two inches from A. Now draw a simple closed curve through these points.

5. Mark some more points on your drawing which are also two inches from the point marked A

6. Are these new points on the picture of the simple closed curve you drew? If not, change your simple closed curve so that these new points will be on it.

7. Does your drawing suggest a simple closed curve which has a special name?

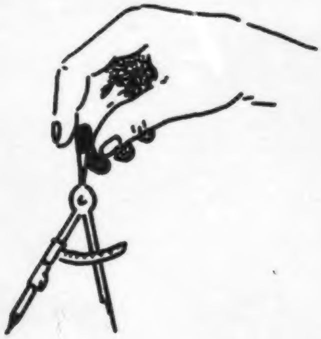
The name of the curve is circle.

A circle cannot be accurately represented by drawing with a pencil and a ruler. A compass is needed.

The easiest way to draw a circle with a compass is to hold the top of the compass between your thumb and index finger. If you press lightly, the compass will work better for you. Press slightly harder on the sharp tip of the compass than you do on the pencil part of the compass.

When you start to draw a circle, do not lift the compass from the paper until the circle is completed. Do not forget to tilt the compass in the direction you are drawing the circle.

Practice using your compass correctly.



8. Directions for drawing a circle.

a) Mark a point on your paper and label it B. B will not be part of the circle.

b) Set your compass so that the metal tip is two inches from the pencil tip.

c) Put the metal tip on the point marked B. Now swing the pencil point so that you draw a simple closed curve. Do not let the distance between the pencil point and metal point change while you are drawing.

You have drawn a picture of a circle.

The point marked B is called the center of the circle. Point B is not part of the circle.

9. Mark two points of your circle. Label the points C and D. Draw a picture of $\overline{BC}$.

B marks the center of the circle and C marks a point on the circle.

$\overline{BC}$ is called a radius of the circle.

10. Draw a picture of $\overline{BD}$ in your drawing. $\overline{BD}$ is also a radius of the circle. Why?

a) Can you draw another radius? If so, do. Call it $\overline{BE}$.

b) Can you draw still another radius? If so, do.

c) How many radii does a circle have? (Radii is the plural of radius.)

11. Are the sentences below true? Use your picture to help you decide.

a) A circle has all its points the same distance from a point inside called the center.

b) B marks the center of this circle.

c) All the radii of a circle have the same length.

d) $\overline{BD}$ is a radius of the circle.

e) $\overline{BC}$ and $\overline{BE}$ are also radii of the circle.

Exercise Set 9

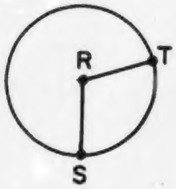
1. a) Mark a point on your paper and label it C. Draw a circle with C as center.

   b) Draw a radius of your circle.

   c) Mark a point of your circle and label it D. Draw $\overline{CD}$.

   d) $\overline{CD}$ is a ______?______ of the circle.

2. Look at this picture.



   a) Name the center of the circle.

   b) Name a radius of the circle.

   c) What is true about the lengths of $\overline{RT}$ and $\overline{RS}$?

3. Mark two points about two inches apart. Call the points A and B.
   - a) Draw a circle with the center at the point A.
   - b) Draw a different circle with point A as the center.
   - c) Draw a third circle with point B as the center.
   - d) Draw a radius of each circle.

4. Mark two points R and S about two inches apart.
   - a) Draw a circle with center at point R and passing through point S.
   - b) Draw a circle with the center at S and passing through R.
   - c) Is $\overline{RS}$ a radius of both circles?

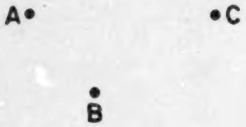
5. Mark two points A and B on your paper.
   - a) Draw a circle with the center at A and having $\overline{AB}$ as a radius.
   - b) Draw three more radii of this circle.

6. Draw two different circles so that a radius of one has the same length as a radius of the other.

7. Draw two different circles so that one has a radius of different length from the other.

8. Draw two different circles with the same center.

9. Trace the points of {A, B, C} on your paper.



a) Draw $\overline{AB}$.

b) Draw the circle with center at A and passing through B.

c) Draw a circle with center at C and a radius equal in length to the length of $\overline{AB}$.

d) Draw a radius of the circle you just made.

e) Is the length of this radius equal to the length of $\overline{AB}$?

10. a) Could the intersection of two circles be the empty set? Draw a figure to show your answer.

b) Could the intersection of two circles be a set with exactly one point? Draw a figure to show your answer.

c) Could the intersection of two circles be a set which has exactly two points? Draw a figure to show your answer.

11. a) Could the intersection of a circle and a line be the empty set? Draw a figure to show your answer.

b) Could the intersection of a circle and a line be a set which has exactly one point? Draw a figure to show your answer.

c) Could the intersection of a circle and a line be a set which has exactly two points? Draw a figure to show your answer.

BRAINTWISTER

12. a) Could the intersection of two circles be a set which has exactly three points?

b) Could the intersection of a circle and a line be a set which has exactly 3 points?

REGIONS IN A PLANE

Working Together

1. Draw a picture of a triangle. Trace the triangle with a blue crayon.

2. Color the part of the plane inside the triangle red. The set of points you colored red is called the interior of the triangle.

3. Color the part of the sheet outside the triangle yellow. This set of points which you colored yellow is part of the exterior of the triangle.

The set of points of the triangle is not in the interior and is not in the exterior of the triangle.

4. Use your compass to draw a circle. Trace the circle with a blue crayon.

5. Color the interior of the circle red.

6. Color the exterior of the circle yellow.

7. Mark a point of the circle. Label it A. Is point A in the interior of the circle? Is A in the exterior of the circle? Mark another point which is not in the interior of the circle and is not in the exterior of the circle.

8. Draw a triangle with blue crayon. Color the interior of the triangle blue also.

9. The part of the plane colored blue is the union of two sets of points. What two sets?

The union of a simple closed curve and its interior is called a plane region. The one you colored blue is called a triangular region.

Exercise Set 10

1. a) Draw a triangle. Color the triangle and its interior red.

   b) What is the name of the part of the plane which is red?

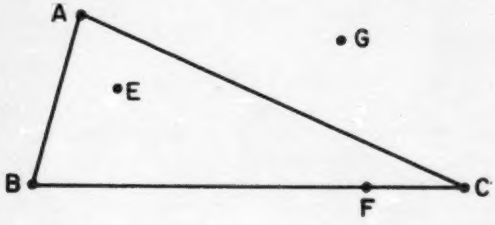
   c) What is the name of the part of the plane which is not red?

2. a) Draw a circle. Color the circle and its interior blue.

   b) What do you think should be the name of the part of the plane which is blue?

   c) What is the name of the part of the plane which is not blue?

3. Look at the figure and the labeled points.



Which sentences are true?

E is: a) a point of the triangle.

b) a point of the interior of the triangle.

c) a point of the exterior of the triangle.

d) a point of the triangular region.

4. F is: a) a point of the triangle.
b) a point of the interior of the triangle.
c) a point of the exterior of the triangle.
d) a point of the triangular region.

5. G is: a) a point of the triangle.
b) a point of the interior of the triangle.
c) a point of the exterior of the triangle.
d) a point of the triangular region.

6. A is: a) a point of the triangle.
b) a point of the interior of the triangle.
c) a point of the exterior of the triangle.
d) a point of the triangular region.

7. Mark a point A and a point B at least two inches from A. Draw a circle with center A and with AB a radius.

Which endings are correct for the figure in Ex. 7?

8. A is a point of

a) the circle.
b) the interior of the circle.
c) the exterior of the circle.
d) the circular region.

9. B is a point of

a) the circle.
b) the interior of the circle.
c) the exterior of the circle.
d) the circular region.

10. Mark a point of the exterior of your circle. Label it C.

11. Mark a point of the circular region. Label it D.

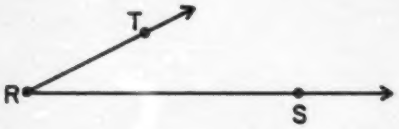
12. Mark a point which is not in the interior and not in the exterior of the circle. Label it E.

Working Together

1. Mark a point R on your paper. Draw a ray with R as endpoint. Mark another point on the ray and label it S.

2. Draw a second ray with R as endpoint. Do not draw it on $\overleftrightarrow{RS}$. Mark a point on this ray and label it T.

Does your drawing look something like this?



This drawing represents a new geometric figure called an angle.

An angle is the union of two rays which have the same endpoint but are not on the same line.

In the figure, R is the vertex of the angle. The endpoint of both rays is called the vertex of the angle.

Each ray is a ray of the angle. $\overrightarrow{RT}$ and $\overrightarrow{RS}$ are rays of the angle in the drawing.

3. Part of an angle is represented by two edges of your desk which meet at a corner.

a) What represents the vertex of the angle?

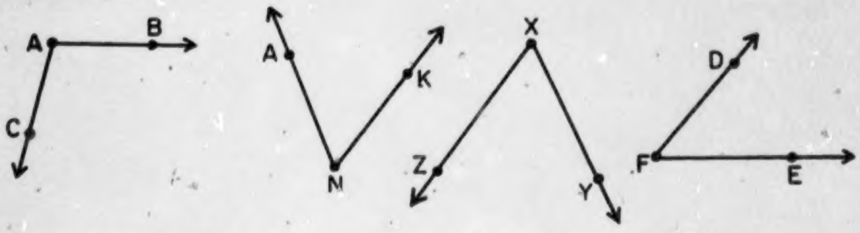
b) What represents the rays of the angle?

c) Why do we say these are only part of the angle?

4. Do the hands of a clock suggest an angle? If so, what represents the vertex? What represents the rays?

5. Describe other things in your classroom which suggest an angle.

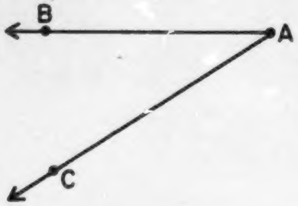
6. In each angle pictured below, name the vertex and the rays.



We name the first angle in the picture $\angle BAC$ or $\angle CAB$. Either is correct. The middle letter must be the label for the vertex.

7. Draw an angle. Label it $\angle SRT$. Did you put the correct letter at the vertex?

8. Below is represented $\angle BAC$. Copy the picture on your paper.



a) Choose a point on $\overrightarrow{AB}$ different from A and B and label it D.

b) Choose a point on $\overrightarrow{AC}$ different from A and C and label it E.

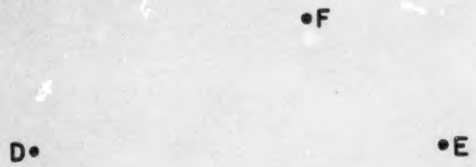
c) Is $\overrightarrow{AB}$ the same ray as $\overrightarrow{AD}$?

d) Is $\overrightarrow{AC}$ the same ray as $\overrightarrow{AE}$?

e) Is $\angle BAC$ the same angle as $\angle DAE$?

No matter how we label an angle, the middle letter always represents the vertex.

9. Three points are shown below.



Write on a sheet of paper the words that complete these sentences.

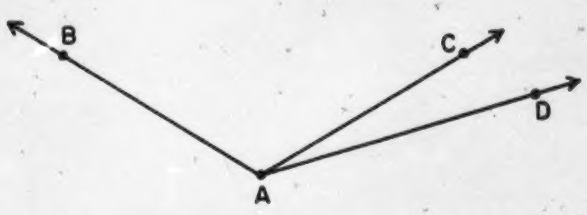
a) There is ____?____ ray through D and F with endpoint F.

b) There is ____?____ ray through F and E with endpoint F.

c) There is ____?____ angle containing D and E with vertex F. This angle is labeled ____?____ or ____?____.

Exercise Set 11

1. Here are three rays. Each has the endpoint A. Name three angles.



2. a) Mark a point C on your paper. Draw a picture of two angles which have the point marked C as a vertex.

   b) Name the rays of each angle.

3. a) Mark a point A on your paper. Draw a picture of at least 4 angles which have the point marked A as a vertex. Do this by drawing 5 different rays, not on the same line, with A as endpoint. Choose a point different from A on each ray. Label these points with the capital letters B, C, D, E, and F.

   b) Name the rays of each angle.

   c) Name each angle.

BRAINTWISTERS

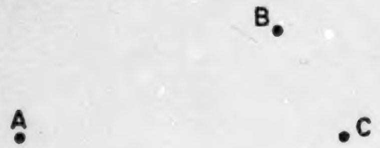
4. Try to repeat Ex. 3 by using only 3 rays (no two of them on the same line) with A as endpoint. Did you get a picture of four angles? How many angles does your picture represent?

5. Try to repeat Ex. 3 by using only 4 rays (no two of them on the same line) with A as endpoint. Did you get a picture of exactly four angles? How many angles does your picture represent?

ANGLES OF A TRIANGLE

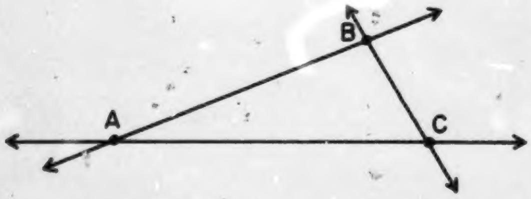
Working Together

1. Look at the points below which are labeled A, B, and C. They are not on the same line. Mark three points like this on your paper and label them.



2. Draw: $\overrightarrow{AB}$
$\overrightarrow{AC}$
$\overrightarrow{BC}$
$\overrightarrow{BA}$
$\overrightarrow{CB}$
$\overrightarrow{CA}$

3. Does your drawing look something like this?



Write on a sheet of paper the words that complete these sentences.

a) $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ form a ____?____.

b) The angle with vertex A which contains B and C is called ____?____.

c) The angle with vertex B which contains A and C is called ____?____.

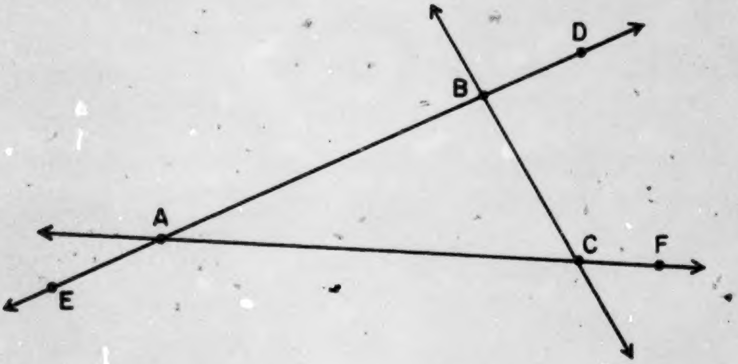
d) The angle with vertex C which contains A and B is called ____?____.

4. Mark a point of $\overrightarrow{AB}$ which is not a point of $\overline{AB}$. Label it D.

5. Mark a point of $\overrightarrow{BA}$ which is not a point of $\overline{AB}$. Label it E.

6. Mark a point of $\overrightarrow{AC}$ which is not a point of $\overline{AC}$. Label it F.

Does your drawing look like this now?



7. Are D, E, and F points of the rays of the angles you named in Ex. 3?

8. a) Are D, E, and F points of the triangle?

b) Are D, E, and F points of the interior of the triangle?

c) Are D, E, and F points of the exterior of the triangle?

Ex. 3 shows that a triangle suggests three angles. These angles are not part of the triangle. This is true because a triangle is made up of segments and an angle is made up of rays.

Remember when we studied circles we spoke of the center of a circle. The center is not part of the circle.

In the same way we say $\angle ABC$, $\angle BCA$, and $\angle CAB$ are angles of the triangle although they are not part of the triangle. We call the vertices of these angles the vertices of the triangle. Vertices is the plural of vertex. The vertices of a triangle are the endpoints of the segments of the triangle.

9. Draw a triangle. Label its vertices D, E, F.

a) Name the three angles of the triangle.

b) The three angles of a triangle suggest how many rays?

Exercise Set 12

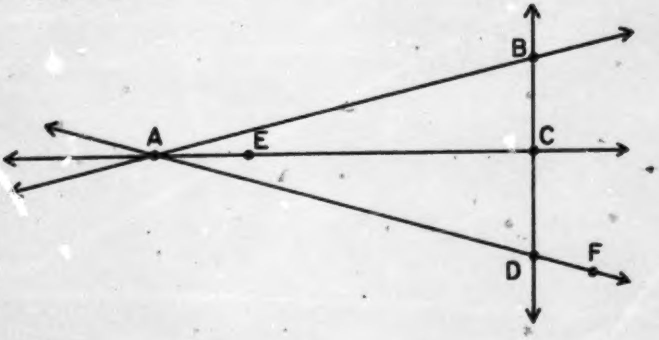
Make drawings to represent

1. A line
2. A ray
3. A segment
4. A simple closed curve

5. A triangle
6. A circle
7. A polygon
8. Two lines which cross
9. A quadrilateral
10. Three lines which cross but not all in the same point
11. An angle
12. The union of a triangle and one angle suggested by the triangle
13. A triangular region

Using the drawing below name:

14. the intersection of $\overrightarrow{AB}$ and $\overrightarrow{DC}$.
15. three different triangles.
16. a segment which is not a side of a triangle.
17. a point of the interior of some triangle.
18. a point of the exterior of triangle ABD.
19. the intersection of $\overleftrightarrow{AF}$ and $\overleftrightarrow{BC}$.
20. the intersection of $\overleftrightarrow{AC}$ and $\overrightarrow{DF}$.
21. the intersection of $\overline{AE}$ and $\overline{BD}$.
22. the endpoint of $\overrightarrow{AE}$.



Which sentences are true?

23. The intersection of two different planes may be:
    a) a line.
    b) the empty set.
    c) a set which has exactly one point.
    d) a plane.

24. The intersection of a line and a plane may be:
    a) a set which has exactly two points.
    b) the empty set.
    c) the line.
    d) the plane.
    e) a set which has exactly one point.

BRAINTWISTERS

25. The intersection of a triangle and a plane may be:
    a) a set which has exactly one point.
    b) the empty set.
    c) the triangle.
    d) a set which has exactly three points.
    e) a set which has more points of the triangle than can be counted but not all the points of the triangle.

26. The intersection of a circle and a plane may be:
    a) a set which has exactly one point.
    b) the empty set.
    c) a set which has exactly two points.
    d) a set which has exactly three points.
    e) the circle.